2020 AMC Mathcounts Countdown

Austin Math Circle

Jan. 2020

Problem 0. In the Welsh village of

Llanfairpwllgwyngyllgogerychwyrndrobwllllantysiliogogogoch,

a merchant is selling plums for the price of five firdlyc apiece. Llywelyn pays three ceiniogau for three plums, and receives a single firdlyc as change. Glyndwr pays six dymeu for two plums, and receives two firdlyc as change. Gwenwynwyn has twelve ceiniogau and sixteen dymeu. How many plums can he buy?

Answer: 19

Problem 1. Jay writes the numbers 1, 2, 3, and 27 on a blackboard, and Josh chooses two distinct integers from the blackboard. Compute the probability that the product of Josh's pair is a perfect cube. Express your answer as a common fraction.

Answer: $\frac{1}{6}$

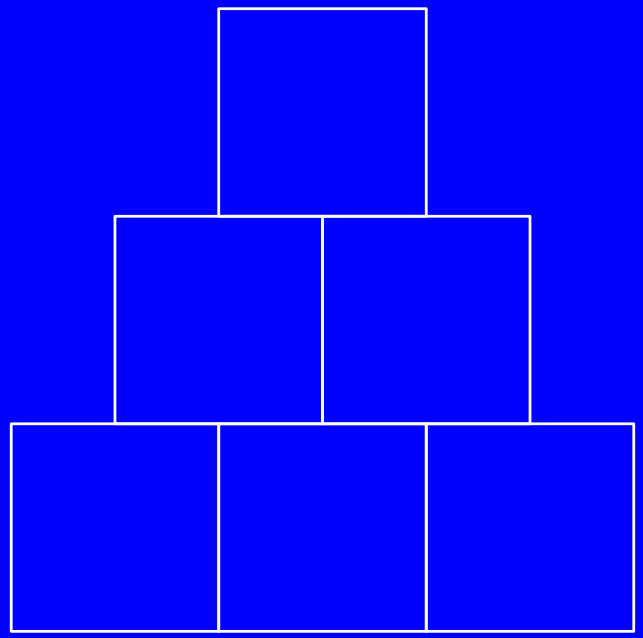
Problem 2. Regular hexagon *AUSTIN* and square *MATH* lie in the same plane and share vertices A and T. If $AU = 2$, what is the area of the union of these two polygons? Express your answer in simplest radical form.

Answer: $16+3\sqrt{3}$ units^2

Problem 3. At 2:57 AM, Heffrey gets off of a bus. After buying 2 bags of candy at $1.59 each, he gets back on at 3:11 AM. How many minutes was Heffrey off of the bus?

Answer: 14 minutes

Problem 4. Ashay has built a pyramid out of six blocks arranged as shown below. He wants to take the blocks away one at a time, but he cannot take a block away while there is another block resting on top of it. In how many ways can Ashay take away the six blocks?



Answer: 16 ways

Problem 5. Rick writes a two-digit integer on the board. Every minute, he replaces the number on the board with its square root, ending when the number is smaller than 2. Compute the largest number of square roots Rick performs.

Answer: 3 square roots

Problem 6. Compute the largest positive integer n such that 2^n divides

$$1! \times 2! \times 3! \times 4! \times 5! \times 6! \times 7! \times 8!$$

Answer: 23

Problem 7. Kelly rolls a fair six-sided die, and Maggie rolls a fair eight-sided die. Compute the probability that the number Kelly rolls is strictly greater than the number Maggie rolls. Express your answer as a common fraction.

Answer: $\frac{5}{16}$

Problem 8. Compute 142857×7.

Answer: 999999

Problem 9. The sleepy sloth wakes up on Monday at 6 AM. Every following day, the sloth wakes up at some random time within an hour of the previous day. On Friday, let the earliest hour the sloth could wake up be m AM and the latest be M AM. Compute $M + m$.

Answer: 12

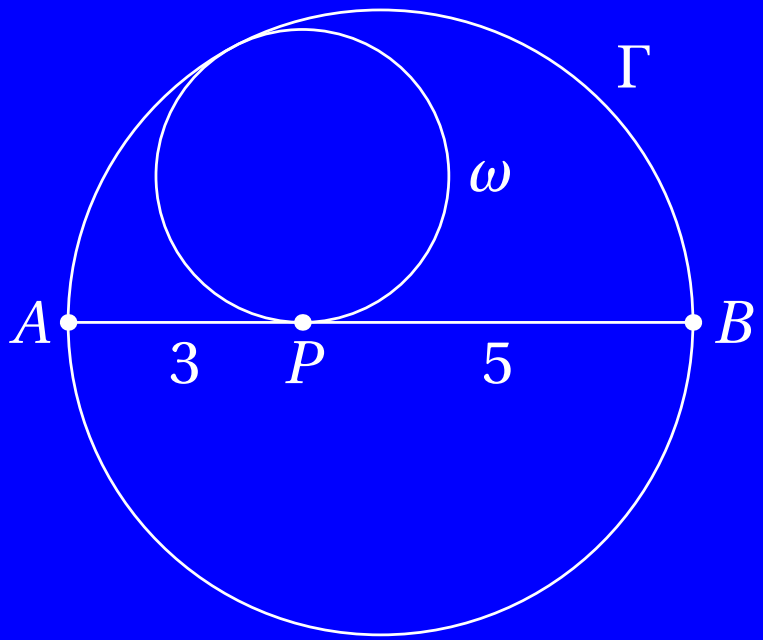
Problem 10. Dylan shuffles a standard 52-card deck and begins drawing cards from it without replacement. The first two cards he draws are both Jacks. What is the probability that the third card he draws is also a Jack? Express your answer as a common fraction.

Answer: $\frac{1}{25}$

Problem 11. Eve chooses a positive integer less than 100 which is divisible by 3. Compute the probability that it is also even. Express your answer as a common fraction.

Answer: $\frac{16}{33}$

Problem 12. Circle Γ has diameter AB. Circle ω is tangent to segment AB at P, and internally tangent to circle Γ. If $AP = 3$ and $BP = 5$, compute the radius of circle ω. Express your answer as a common fraction.



Answer: $\frac{15}{8}$ units

Problem 13. Ronda Rousey replaces rosy rocks with Reckless Rick's red rings. Reckless Rick reports 10 red rings, but Ronda Rousey can only replace Reckless Rick's red rings with her rosy rocks at a rate of 3 or fewer red rings for 1 rosy rock. Once all red rings are removed, compute the minimum possible number of rosy rocks at the end.

Answer: 4 rosy rocks

Problem 14. If a and b are positive real numbers satisfying

$$a - b = 3$$

$$a^2 - b = 33$$

then compute $a^3 - b$.

Answer: 213

Problem 15. Alicia rolls a fair twelve-sided die, and Sophie rolls a fair twenty-sided die. Compute the probability that the numbers showing on both dice are prime. Express your answer as a common fraction.

Answer: $\frac{1}{6}$

Problem 16. What is the sum of all the three-digit positive integers with the property that each of their digits is either 1, 3, or 6?

Answer: 9990

Problem 17. If x, y are real numbers such that $(x+1)(y+1) = 20$ and $(x+2)(y+2) = 40$, compute $(x+3)(y+3)$.

Answer: 62

Problem 18. How many positive integers less than 30 can be written as the product of two distinct primes?

Answer: 7 integers

Problem 19. Tom writes the numbers 1, 2, 2^2, 2^3, 2^4 and 2^5 on a blackboard. He then selects two and looks at their difference. Compute the number of possible positive differences.

Answer: 15 differences

Problem 20. Matthew has one coin, and Nir has no coins. Rich Farmer Joe will continually give coins randomly to either Matthew or Nir until the total number of coins Matthew and Nir have is divisible by 5. Compute the expected number of coins Matthew gains.

Answer: 2 coins

Problem 21. Matthew's favorite primes are $3, 7, 13$, and 17. He randomly chooses two different primes out of these four and sums their cubes. Find the probability that this sum is divisible by 8.

Answer: $\frac{1}{3}$

Problem 22. A rhombus has perimeter 20 and area 11. Compute the sum of its diagonals.

Answer: 12 units

Problem 23. Regular hexagon *CLAIRE* has area 1. Compute the area of the intersection of triangle *AIR* and quadrilateral *ERIC*. Express your answer as a common fraction.

Answer: $\frac{1}{12}$ units^2

Problem 24. Josh is buying coke and mentos at a store. He pays the cashier $15, including both the cost and some amount of change. However, the cashier is very tired, and inadvertently swaps the dollar and cent amounts for the purchase. He then proceeds to give Josh back half the change he needs. What was the original price of the coke and mentos?

Answer: $5.10

Problem 25. A point on the graph of $y = x^2$, other than the origin, is twice as far from the x-axis as it is from the y-axis. How far away is it from the origin? Express your answer in simplest radical form.

Answer: $2\sqrt{5}$ units

Problem 26. Compute the largest positive integer that is equal to four times the sum of its digits.

Answer: 48

Problem 27. If

$$x - y = 7$$

$$x^2 - y^2 = 91$$

then compute x.

Answer: 10

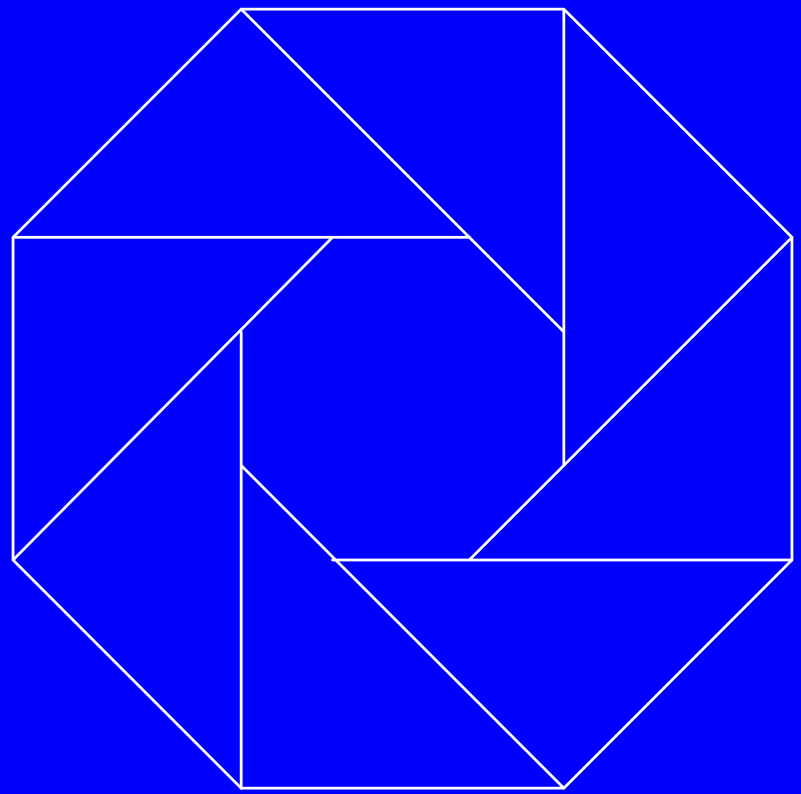
Problem 28. How many distinct positive integers divide $4^{(4^4)}$?

Answer: 513 integers

Problem 29. Wainright writes the number 123456 on a blackboard, and then Eve erases some (possibly zero) digits uniformly at random, so that all possible combinations of digits Eve could have erased are equally likely. Compute the probability that Eve's erasing leaves behind a nonzero number of digits that form an even integer.

Answer: $\frac{21}{32}$

Problem 30. The interior of a large regular octagon is partitioned as shown into eight isosceles right triangles and one small regular octagon. Compute the ratio of the area of the large regular octagon to the area of the small regular octagon. Express your answer in simplest radical form.



Answer: $3+2\sqrt{2}$

Problem 31. A four-digit year $\underline{A}\,\underline{B}\,\underline{C}\,\underline{D}$ is called *powerful* if the product $\underline{A}\,\underline{B} \times \underline{C}\,\underline{D}$ is a perfect square. For example, 2020 is a powerful year since $20 \times 20 = 400$ is a perfect square. What is the next powerful year after 2020?

Answer: 2045

Problem 32. Find the unique integer between $\frac{2020}{39}$ and $\frac{3180}{61}$.

Answer: 52

Problem 33. Compute the sum of all positive integers n less than or equal to 27 for which 27 divides $n^2 - 9$.

Answer: 81

Problem 34. A shirt costs \$10. The price of the shirt is raised by 30%, and then later it is lowered by 30%. Afterwards, how much does the shirt cost?

Answer: $9.10

Problem 35. In the year 2020, there are two months such that the thirteenth day of the month falls on a Friday. One of these months is March. What is the other one?

Answer: November

Problem 36. Compute the sum of all distinct prime divisors of $2020^{20+20} \times (20+20)^{2020}$.

Answer: 108

Problem 37. Harrison has two pizzas, a small 8-inch diameter pizza that is cut into 8 congruent slices, and a large 12-inch diameter pizza that is cut into 10 congruent slices. Compute the ratio of the area of a slice of the large pizza to the area of a slice of the small pizza. Express your answer as a common fraction.

Answer: $\frac{9}{5}$

Problem 38. Compute

$$1024 - 512 + 256 - 128 + 64 - 32 + 16 - 8 + 4 - 2 + 1.$$

Answer: 683

Problem 39. Major Tom is orbiting the Earth at an altitude of 450 miles. Assuming the Earth is a perfect sphere with radius 4000 miles, compute the largest possible distance (in miles) between Major Tom and any point on Earth's surface that he can see from his current location.

Answer: 1950 miles

Problem 40. Alice tells Bob that the product of two distinct positive integers is 49, but before she tells him their sum, Bob interrupts her and says the sum for her. What is the sum?

Answer: 50

Problem 41. A cube has surface area 42. What is its volume? Express your answer in simplest radical form.

Answer: $7\sqrt{7}$ units^3

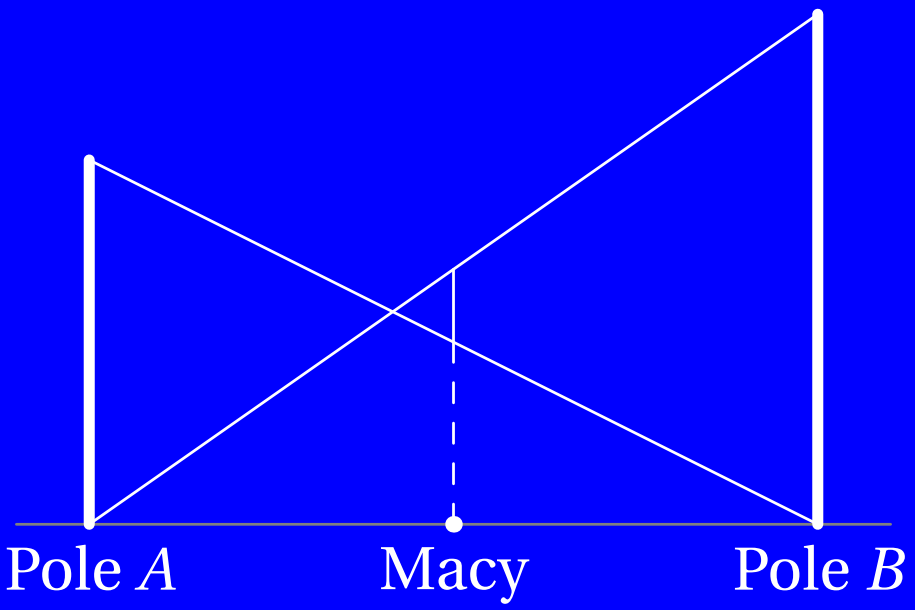
Problem 42. Alice chooses a random set of three distinct nonzero digits, and Eve arranges them into a three-digit integer. Compute the probability that Eve cannot form an even integer. Express your answer as a common fraction.

Answer: $\frac{5}{42}$

Problem 43. Sunshine flips a fair coin six times. What is the probability that she sees strictly more heads than tails?

Answer: $\frac{11}{32}$

Problem 44. Pole A of height 10 feet and pole B of height 14 feet are stuck in the ground. A cord connects the top of pole A to the bottom of pole B, and another cord connects the top of pole B to the bottom of pole A, as shown. Macy stands at the midpoint between poles A and B and looks straight up. What is the vertical distance in feet between the two cords there?



Answer: 2 feet

Problem 45. Given that $3x = 6$, $2x + 2y = 14$, and $x + y + z = 28$, find z.

Answer: 21

Problem 46. For real x and y,

$$y^3 = 3x^2 + 3x + 1$$

$$x^3 = 3y^2 + 3y + 1$$

Find $\frac{3x+y}{3y+x}$.

Answer: 1

Problem 47. Delaney writes the integers “0, 0, 0” on a board, and then her two friends each randomly add 1 to one of Delaney’s integers. So, the board could read “0, 2, 0” or “1, 0, 1” at the end. Compute the probability the number 1 is on Delaney’s board at the end. Express your answer as a common fraction.

Answer: $\frac{2}{3}$

Problem 48. Garrett has a bag that contains four red marbles and four blue marbles. He begins to draw marbles from the bag, one at a time, without replacement, and writes down the color of each marble he drew in order. How many different possible sequences of colors could Garrett have written down after he has drawn six of the eight marbles?

Answer: 50 sequences

Problem 49. Let $n = 1234$. Compute $\lfloor \sqrt{n+1} \rfloor - \lfloor \sqrt{n} \rfloor$. *($\lfloor x \rfloor$ denotes the greatest integer that is less than or equal to x.)*

Answer: 0

Problem 50. A squirrel draws 3 circles of radius 1 with their centers at the three vertices of an equilateral triangle with side length 1. A fourth circle is drawn such that the three smaller circles are all inside it and tangent to it. Compute the radius of the fourth circle, expressed as a common fraction in simplest radical form.

Answer: $\frac{3+\sqrt{3}}{3}$ units

Problem 51. Let $a \# \# b = 2$ if $a - b$ is a cube but not a square, 1 if $a - b$ is a square, and 0 otherwise. Find

$$(100 \# \# 1) + (100 \# \# 2) + \cdots + (100 \# \# 100).$$

Answer: 14

Problem 52. If $a \parallel b = \frac{ab}{a+b}$, compute

$$\left(\left(1 \parallel \frac{1}{3}\right) \parallel \frac{1}{8}\right) \parallel \frac{1}{5}$$

Express your answer as a common fraction.

Answer: $\frac{1}{17}$

Problem 53. Miss Lilly wants to distribute four identical granola bars among her four students such that no student receives more than two granola bars. In how many ways can she do this?

Answer: 19 ways

Problem 54. Let $A = (-4, 0)$, $B = (0, -5)$, and $C = (4, 5)$. Compute the area of triangle ABC.

Answer: 30 units^2

Problem 55. If x, y are positive real numbers such that $x^2 + y^2 = 74$ and $2x^2 + y^2 = 99$, find $x + y$.

Answer: 12

Problem 56. Rectangle $RICH$ has $RI = 12$ and $IC = 5$. Compute the area of the set of points in the interior of rectangle $RICH$ that are closer to side $\overline{RI}$ than to any other side of the rectangle. Express your answer as a common fraction.

Answer: $\frac{95}{4}$ units^2

Problem 57. Find the smallest prime factor of $\frac{15^3-1}{2}$.

Answer: 7

Problem 58. Compute the number of six-digit integers whose digit sum is divisible by 10.

Answer: 90000 integers

Problem 59. The River City High School marching band has 76 students who play the trombone and 110 students who play the cornet. If every student plays at least one of these two instruments, and exactly half of the students in the band play both, compute the total number of students in the River City High School marching band.

Answer: 124 students

Problem 60. Physicist Phoebe is gluing spherical cows together. Whenever she sees two cows of the same size, she glues them together into one bigger cow, whose size is the sum of the sizes of the two smaller cows. For example, if she has ten size-1 cows, Phoebe would glue them together into five larger size-2 cows, and eventually one size-8 cows with one size-2 cows. What is the minimum number of cows Phoebe must have started with if she leaves four cows of different sizes at the end? (All cows are initially size-1 cows.)

Answer: 15 cows