MATHCOUNTS®

2018 Austin Math Circle Competition
Sprint Round
Problems 1 – 30

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HONOR PLEDGE

I pledge to uphold the highest principles of honesty and integrity as a Mathlete®. I will neither give nor accept unauthorized assistance of any kind. I will not copy another's work and submit it as my own. I understand that any competitor found to be in violation of this honor pledge is subject to disqualification.

Signature: _____ Date: _____

Printed Name: _____

School: _____

DO NOT BEGIN UNTIL YOU ARE INSTRUCTED TO DO SO

This round of the competition consists of 30 problems. You will have 40 minutes to complete all the problems. You are not allowed to use calculators or other aids during this round. If you are wearing a calculator wrist watch, please give it to your proctor now. Calculations may be done on scratch paper. All answers must be complete, legible and simplified to lowest terms. Record only final answers in the blanks in the left-hand column of the competition booklet. If you complete the problems before time is called, use the remaining time to check your answer.

In each written round of the competition, the required unit for the answer is included in the answer blank. The plural form of the unit is always used, even if the answer appears to require the singular form of the unit. The unit provided in the answer blank is the only form of the answer that will be accepted.

Total Correct	Scorer's Initials

1. _____ What is the difference between four dozens and three half-dozens?

2. _____ What is the sum of the two-digit numbers that are multiples of five but not of two or three?

3. _____ At a store, a certain coat costs $100 in November. In December, the store puts the coat on sale for 25% off. When the sale ends in January, the discount price is marked up by 30%. What is the price of the coat at the end of January?

4. _____ A *Metro News One* forecast predicts a 50% chance of rain on Monday and a 40% chance of rain on Tuesday. Find the probability that it will rain on Monday, Tuesday, or both days. Express your answer as a percentage.

5. _____ Maggie scored an average of 88 points on five quizzes. What score would she need to earn on her sixth quiz to raise her average to 90?

6. _____ The seven points *A*, *B*, *C*, *D*, *E*, *F*, and *G* are evenly spaced in alphabetical order on segment *AG*, which has length 210. Find the length of segment *CF*.

7. _____ Barbara earned the following scores on eight math tests: 94, 86, 97, 95, 92, 85, 93, and 99. What is the median of Barbara's test scores? Express your answer as a decimal to the nearest tenth.

8. _____ In a technology club, every student likes math or science. 19 of the students like math, 22 like science, and 7 like both. How many students are in the club?

9. _____ A certain math test has fifteen questions. The answer to each question is a positive integer. All of the questions except question #8 have the same answer, which is an even integer, and the sum of the answers to all fifteen questions is 586. What is the smallest possible answer to question #8?

10. _____ What is the largest possible number that can appear in a list of five unique integers with median five and range five?

11. _____ Jim writes the expression 1 + 2 + 3 + 4 + 5 on a blackboard. Then, Dwight erases two of the addition symbols and replaces them with other standard arithmetic operations: −, *, and ÷. After that, Michael inserts a single set of parentheses into the expression. What is the smallest possible value of the new expression? For example, one possible final expression is 1 − 2 + (3 + 4 ÷ 5).

12. _____ Let $f^1(x) = x^2 - 1$, and let $f^n(x) = f^1(f^{n-1}(x))$ for all n > 1. Find $f^{100}(1)$.

13. _____ A wooden cube of side-length one is cut into two pieces by a plane parallel to one of its faces. What is the sum of the surface areas of the two pieces?

14. _____ Chord *AB* of length 20 is drawn in a circle with center *O* and radius 26. Find the distance from *O* to the midpoint of *AB*.

15. _____ Two standard fair six-sided dice are rolled. What is the probability that the sum of the numbers on top of the two dice is prime? Express your answer as a common fraction.

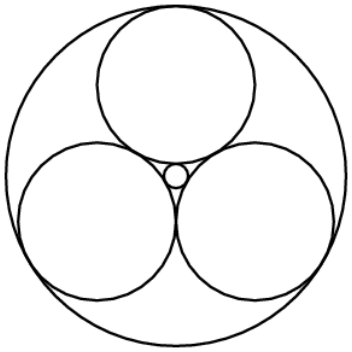
16. _____ Find the positive real number x satisfying $2^{x^2} = 4^{x^4}$. Express your answer as a common fraction.

17. _____ Find the sum of the squares of the values of x satisfying $x^3 - ax^2 + bx - c = 0$. Express your answer in terms of $a, b,$ and c.

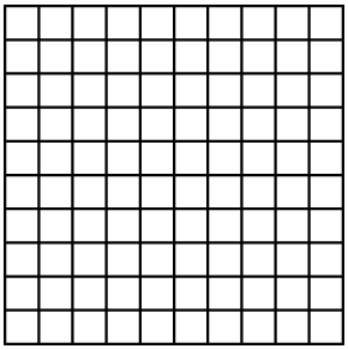
18. _____ Let point *A* be (0, 0), and let point *B* be (6, 0). Point *C* is randomly selected on the line segment connecting (0, 3) and (6, 3). What is the expected value of the area of triangle *ABC*?

19. _____ For how many positive integers n at most 100 does the decimal representation of $\frac{1}{n}$ terminate?

20. _____ Three circles of radius one are drawn on a sheet of paper such that each is externally tangent to the other two. Chris draws two more circles on the paper. One of these circles is externally tangent to the former three circles while the other is internally tangent to the circles. What is the product of the areas of these two new circles? Express your answer as a common fraction in terms of π.



21. _____ How many rectangles of any size have their sides contained in the edges of the ten-by-ten grid of unit squares shown below?



22. _____ What is the hundreds digit of 11^{2018}?

23. _____ A car has a faulty odometer with a display that resets back to zero when it reaches a certain number of miles. At first, the odometer reads 215 miles. Then, after the car is driven for a while, the odometer has reset once and finally reads 422. When the car is driven the same length again, the odometer resets twice more and eventually reads 56. At what number of miles does the odometer reset?

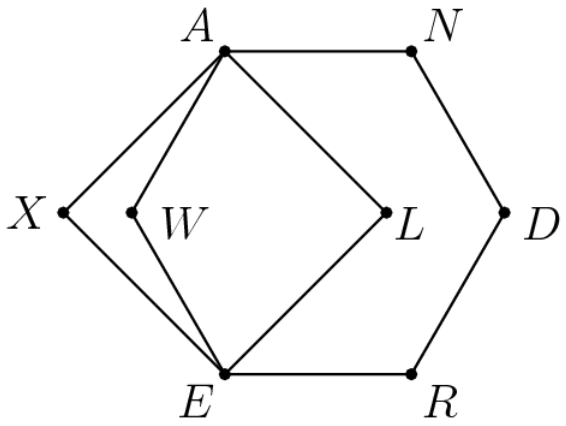
24. _____ The permutations of the letters in the word *CIRCLE* are listed in alphabetical order. In what position will the word *CLERIC* be listed? For example, the permutation *CCEILR* appears in position one while *CCEIRL* appears in position two.

25. _____ Let p be the probability that when a coin is flipped n times, no two consecutive flips show heads. Find the value of n that minimizes $\left|p - \frac{1}{3}\right|$.

26. _____ Compute $1^2 + 3^2 + 5^2 + \cdots + 49^2$.

27. _____ Albert and Betty each pick a digit (from zero to nine), but they don't tell each other which digit they picked. When their numbers are multiplied together, the result's final digit is x. Albert says, "As far as I can tell, x could be any of the ten digits." Betty responds, "I wasn't sure before, but now I know what x is." Find x, assuming Albert and Betty have both used correct logic.

28. _____ Regular hexagon *ANDREW* and square *ALEX* lie in the same plane, as shown in the figure. What fraction of the area of *ANDREW* is also inside of *ALEX*? Express your answer as a common fraction in simplest radical form.



29. _____ Paul chooses an integer r uniformly at random from the set $\{0, 1, 2, 3, 4\}$. He then places r red balls and 4-r blue balls into an urn. Amelia draws a ball from the urn, sees that it is red, and places it back in the urn. Afterwards, Claire draws a ball from the urn, sees that it also is red, and places it back in the urn. Now, Susan is about to draw a ball from the urn. What is the probability that it will be red? Express your answer as a common fraction.

30. _____ The number 104060405 is the product of four distinct prime numbers, all of which are less than 10000. What is the largest of these four primes?

Forms of Answers

The following list explains acceptable forms for answers. Coaches should ensure that Mathletes are familiar with these rules prior to participating at any level of competition. Judges will score competition answers in compliance with these rules for forms of answers.

Units of measurement are not required in answers, but they must be correct if given. When a problem asks for an answer expressed in a specific unit of measure or when a unit of measure is provided in the answer blank, equivalent answers expressed in other units are not acceptable. For example, if a problem asks for the number of ounces and 36 oz is the correct answer, 2 lbs 4 oz will not be accepted. If a problem asks for the number of cents and 25 cents is the correct answer, $0.25 will not be accepted.

All answers must be expressed in simplest form. A "common fraction" is to be considered a fraction in the form $\pm\frac{a}{b}$, where a and b are natural numbers and GCF(a, b) = 1. In some cases the term "common fraction" is to be considered a fraction in the form $\frac{A}{B}$, where A and B are algebraic expressions and A and B do not share a common factor. A simplified "mixed number" ("mixed numeral," "mixed fraction") is to be considered a fraction in the form $\pm N\frac{a}{b}$, where N, a and b are natural numbers, $a < b$ and GCF(a, b) = 1. Examples:

Problem: What is 8 ÷ 12 expressed as a common fraction? *Answer:* $\frac{2}{3}$ *Unacceptable:* $\frac{4}{6}$

Problem: What is 12 ÷ 8 expressed as a common fraction? *Answer:* $\frac{3}{2}$ *Unacceptable:* $\frac{12}{8}$, $1\frac{1}{2}$

Problem: What is the sum of the lengths of the radius and the circumference of a circle with diameter $\frac{1}{4}$ unit expressed as a common fraction in terms of π? *Answer:* $\frac{1+2\pi}{8}$

Problem: What is 20 ÷ 12 expressed as a mixed number? *Answer:* $1\frac{2}{3}$ *Unacceptable:* $1\frac{8}{12}$, $\frac{5}{3}$

Ratios should be expressed as simplified common fractions unless otherwise specified. Examples:
Simplified, Acceptable Forms: $\frac{7}{2}$, $\frac{3}{\pi}$, $\frac{4-\pi}{6}$ *Unacceptable:* $3\frac{1}{2}$, $\frac{\frac{1}{4}}{3}$, 3.5, 2:1

Radicals must be simplified. A simplified radical must satisfy: 1) no radicands have a factor which possesses the root indicated by the index; 2) no radicands contain fractions; and 3) no radicals appear in the denominator of a fraction. Numbers with fractional exponents are *not* in radical form. Examples:
Problem: What is the value of $\sqrt{15}\times\sqrt{5}$? *Answer:* $5\sqrt{3}$ *Unacceptable:* $\sqrt{75}$

Answers to problems asking for a response in the form of a dollar amount or an unspecified monetary unit (e.g., "How many dollars...," "How much will it cost...," "What is the amount of interest...") should be expressed in the form ($) *a.bc*, where *a* is an integer and *b* and *c* are digits. The *only* exceptions to this rule are when a is zero, in which case it may be omitted, or when b and c both are zero, in which case they both may be omitted. Answers in the form ($)*a.bc* should be rounded to the nearest cent unless otherwise specified. Examples:
Acceptable: 2.35, 0.38, .38, 5.00, 5 *Unacceptable:* 4.9, 8.0

Do not make approximations for numbers (e.g., π, $\frac{2}{3}$, $5\sqrt{3}$) in the data given or in solutions unless the problem says to do so.

Do not perform any intermediate rounding (other than the "rounding" a calculator does) when calculating solutions. All rounding should be done at the end of the computation process.

Scientific notation should be expressed in the form $a \times 10^n$ where a is a decimal, $1 \le |a| < 10$, and n is an integer. Examples:
Problem: What is 6895 expressed in scientific notation? *Answer:* 6.895×10^3
Problem: What is 40,000 expressed in scientific notation? *Answer:* 4×10^4 *or* 4.0×10^4

An answer expressed to a greater or lesser degree of accuracy than called for in the problem will not be accepted. Whole number answers should be expressed in their whole number form.
Thus, 25.0 will not be accepted for 25, and 25 will not be accepted for 25.0.

The plural form of the units will always be provided in the answer blank, even if the answer appears to require the singular form of the units.

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2018 Austin Math Circle Competition
Target Round
Problems 1 and 2

Name _____

School _____

DO NOT BEGIN UNTIL YOU ARE INSTRUCTED TO DO SO.

This section of the competition consists of eight problems, which will be presented in pairs. Work on one pair of problems will be completed and answers will be collected before the next pair is distributed. The time limit for each pair of problems is six minutes. The first pair of problems is on the other side of this sheet. When told to do so, turn the page over and begin working. This round assumes the use of calculators, and calculations also may be done on scratch paper, but no other aids are allowed. All answers must be complete, legible and simplified to lowest terms. Record only final answers in the blanks in the left-hand column of the problem sheets. If you complete the problems before time is called, use the time remaining to check your answers.

Problem 1	Problem 2	Scorer's Initials

1. _____ Joe is counting by threes. The first number he says is 100, and the second number he says is 103. Find the hundredth number he says.

2. _____ A truck starts with 15 gallons of fuel in its tank and drives 50 miles. Then, 10 more gallons of fuel are put into the tank and the truck drives 42 miles. Finally, 18 more gallons of fuel are added and the truck drives another 60 miles. What is the minimum fuel efficiency, in miles per gallon, that would allow the truck to complete this process without ever running out of fuel? Express your answer as a decimal to the nearest hundredth.

MATHCOUNTS®

2018 Austin Math Circle Competition
Target Round
Problems 3 and 4

Name _____

School _____

DO NOT BEGIN UNTIL YOU ARE INSTRUCTED TO DO SO.

Problem 3	Problem 4	Scorer's Initials

3. _____ A certain company runs a two-digit number of factories that each produce a greater two-digit number of widgets per day. In total, the company produces 1547 widgets per day. Find the number of factories the company runs.

4. _____ Richard has an unfair coin that comes up heads when flipped with probability p. If he flips the coin six times, the probability that it comes up heads exactly three times is 0.02. Find the largest possible value of p. Express your answer as a common fraction in simplest radical form.

MATHCOUNTS®

2018 Austin Math Circle Competition
Target Round
Problems 5 and 6

Name _____

School _____

DO NOT BEGIN UNTIL YOU ARE INSTRUCTED TO DO SO.

Problem 3	Problem 4	Scorer's Initials

5. _____ Find the area of the figure in the Cartesian coordinate plane bounded by the circle with center (0, 0) and radius 4 and above the lines $y = 0$ and $y = x$. Express your answer in terms of π.

6. _____ Eight teams are competing in a single elimination basketball tournament. Every team is assigned a unique integer rank from one to eight. In every round of the tournament, each team is randomly assigned to an opponent and plays a match against them. The winning team advances to the next round and the losing team is eliminated. In every game, the team with the numerically lower rank is the winner. Find the expected value of the sum of the ranks of the teams that play against the team with rank one. Express your answer as a common fraction.

MATHCOUNTS®

2018 Austin Math Circle Competition
Target Round
Problems 7 and 8

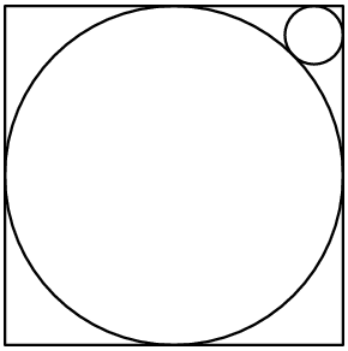
Name _____

School _____

DO NOT BEGIN UNTIL YOU ARE INSTRUCTED TO DO SO.

Problem 3	Problem 4	Scorer's Initials

7. _____ A circle is inscribed in a square, and another circle is externally tangent to the first circle and tangent to two sides of the square, as shown in the figure below. What is the ratio of the area of the large circle to the area of the small circle? Express your answer in simplest radical form.



8. _____ Tomas writes two three-digit positive base-ten integers (without leading zeros), one of which is three times the other, on a blackboard. Later, Ben sees the blackboard, but he assumes the two integers on it are written in base 16. Under this assumption, he converts the two integers to base 10 and divides the larger one by the smaller one. What is the difference between the largest and smallest possible values for Ben's result? Express your answer as a common fraction.

MATHCOUNTS®

2018 Austin Math Circle Competition
Team Round
Problems 1 – 10

School _____

Team Members_____ , Captain

_____

_____

_____

DO NOT BEGIN UNTIL YOU ARE INSTRUCTED TO DO SO.

This section of the competition consists of 10 problems which the team has 20 minutes to complete. Team members may work together in any way to solve the problems. Team members may talk during this section of the competition. This round assumes the use calculators, and calculations may also be done on scratch paper, but no other aids are allowed. All answers must be complete, legible, and simplified to lowest terms. The team captain must record the team's official answers on his/her own problem sheet, which is the only sheet that will be scored.

Total Correct	Scorer's Initials

1. _____ A perfect number is a number that is equal to the sum of its factors, not including itself. The third-smallest perfect number is 496. How many factors does it have (including itself)?

2. _____ Ted takes out a $1000 loan from a local bank. The first year, it accumulates 6% interest. After two years, Ted owes $1113. Find the interest rate Ted paid during the second year. Assume that Ted does not pay off any of his debt and that interest is compounded (i.e. the first year's interest accumulates interest during the second year). Express your answer as a percentage.

3. _____ A rectangular tablet has a diagonal of length seven inches, while a similar rectangular tablet has a diagonal of length ten inches. What is the ratio of the area of the second tablet to that of the first tablet? Express your answer as a common fraction. Two rectangles are similar if their side-lengths are in the same ratio.

4. _____ Eric randomly picks a positive integer n from 1 to 1000. What is the expected value of the number of digits in the decimal representation of n? Express your answer as a decimal to the nearest thousandth.

5. _____ A frog hops around the perimeter of a rectangular lake from one corner to the opposite corner. At the same time, a toad, starting at the same corner as the frog, swims directly across the lake to the opposite corner. The frog and the toad reach the opposite corner of the lake at the same time. If the speed at which the frog hops is 40% faster than the speed at which the toad swims, what is the ratio of the length of the longer side of the lake to the length of its shorter side? Express your answer as a common fraction.

6. _____ A spider is located at one vertex of a rectangular prism with sides of lengths 1, 2, and 3, and a fly is situated on the vertex furthest from the spider. What is the length of the shortest path over the surface of the prism that the spider can take to catch the fly? Express your answer in simplest radical form.

7. _____ What is the least positive integer n such that the value $\frac{9009}{n}$ does not terminate with a finite number of digits when written as a decimal?

8. _____ Wolstenholme and Zsigmondy compete head-to-head to determine which of the two is a better mathematician. They are given a single number theory problem to solve, and both solve it in a two-digit integer number of seconds. Afterwards, they observe that:

- The product of their times is a perfect square.
- The sum of their times is a prime number.
- When Wolstenholme's time w and Zsigmondy's time z are concatenated, the result $100w + z$ is a perfect square.

How many seconds passed before the first person solved the problem?

9. _____ A "time-number" is a six-digit number ***abcdef*** (i.e. with units digit *f*, tens digit *e*, etc) such that ***ab*** is an integer, potentially written with leading zeroes, between zero and twenty-three and ***cd*** and ***ef*** are both integers, possibly written with leading zeroes, between zero and fifty-nine. For example, 123456, 235959, and 000000 are time-numbers, but 999999 is not. How many time-numbers ***abcdef*** exist such that ***abcdef*** is a palindrome but ***abcd*** is not?

10. _____ An operation ♣ maps every pair of positive real numbers x, y to a positive real number $x \clubsuit y$. The following identities hold for this operation over all pairs of positive real numbers x, y:

- $x \clubsuit y = y \clubsuit x$
- $x \clubsuit x = \frac{x}{2}$
- $2x \clubsuit 2y = 2(x \clubsuit y)$
- $\frac{x \clubsuit (y+1)}{y \clubsuit (x+1)} = \frac{x(y+1)}{y(x+1)}$

Find the value of $20 \clubsuit 17$. Express your answer as a common fraction.

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The plural form of the units will always be provided in the answer blank, even if the answer appears to require the singular form of the units.

Countdown Round Problem One

Let $f(x) = 10x + 7$. Compute $f(f(f(f(f(1)))))$.

Countdown Round Problem Two

Janet buys a bag of jellybeans. She eats 30% of them, loses 8 of them, eats 20% of what remains, shares 24 with her friends, eats 70% of the remaining ones, and gives the last 6 to her younger brother. How many jellybeans were in the bag when she bought it?

Countdown Round Problem Three

How many elements do the sequences 4, 7, 10, . . . 100 and 4, 8, 12, . . . 100 share?

Countdown Round Problem Four

What is the least prime number that can be written as the sum of three distinct prime numbers?

Countdown Round Problem Five

A hare jumps at a rate of 100 units per second, but after every ten seconds of jumping it takes a five-second break to cool off. A tortoise crawls along the same path at a rate of 80 units per second without breaks. After how many seconds will the tortoise first be 1500 units ahead of the hare? Express your answer as a decimal to the nearest hundredth.

Tiebreaker Problem

How many permutations a_1, a_2, a_3, a_4, a_5 of the positive integers one through five exist such that for any non-empty proper subset of the permutation, the sum of the values of the a_i in the subset is not equal to the sum of the values of i in the subset?

For example, the sequence $3, 2, 1, 5, 4$ does not satisfy this condition, because $a_4 + a_5 = 4 + 5$. However, $5, 1, 2, 3, 4$ is satisfactory.

If your answer is *a* and the correct answer is *c*, your tiebreaker score will be the ratio of the greater of *a* and *c* to the lesser. The student with the lowest score will have the tie broken in his or her favor.

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2018 Austin Math Circle Competition
Sprint Round
Answers

1. 30
2. 360
3. $97.50
4. 70%
5. 100
6. 105
7. 93.5
8. 34
9. 26
10. 8
11. -44
12. -1
13. 8
14. 24
15. 5/12
16. sqrt(2)/2
17. a^2-2b
18. 9
19. 15
20. pi^2/9
21. 3025
22. 4
23. 573
24. 84
25. 6
26. 20825
27. 5
28. (1+sqrt(3))/6
29. 5/6
30. 2081

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2018 Austin Math Circle Competition
Target Round
Answers

1. 397
2. 3.68
3. 17
4. (5+sqrt(15))/10
5. 6pi
6. 353/60
7. 17+12sqrt(2)
8. 204/359

MATHCOUNTS®

■ Test Name ■
Team Round
Answers

1. 10

2. 5%

3. 100/49

4. 2.893

5. 4/3

6. 3sqrt(2)

7. 17

8. 16

9. 93

10. 340/37