MATHCOUNTS

■ 2021 Austin Math Circle (Online) Practice Competition ■
Sprint Round
Problems 1 – 30

DO NOT BEGIN UNTIL YOU ARE INSTRUCTED TO DO SO

This section of the competition consists of 30 problems. You will have 40 minutes to complete all the problems. You are not allowed to use calculators, books, or other aids during this round. Calculations may be done on scratch paper. All answers are integers, and units should not be included as part of your answer. Record only final answers in the Google form provided. If you complete the problems before time is called, use the remaining time to check your answers.

In each written round of the competition, the required unit for the answer is clarified in the problem statement, although it is not written in the answer blank in the Google form. The unit asked for in the problem statement is the only form of the answer that will be accepted.

1. How many positive integers strictly less than 100 are either prime or composite, but not both?

2. If 640 acres is a square mile, and a mile is 1600 meters, how many square meters is an acre?

3. Anastasia and Bananastasia are driving through a circular roundabout. Anastasia starts at one point of the circle, and travels at a constant speed along the circumference counterclockwise to the point diametrically opposite her starting point. Meanwhile, Bananastasia starts at the same point as Anastasia, but drives straight across the roundabout directly to the other side, also at a constant speed. If they start at the same time and finish at the same time, Anastasia must be traveling x times as fast as Bananastasia. Compute the nearest integer to $10x$.

4. Gordon Ramsay and Chef Rush are in a cooking contest which consists of a number of matches. If Rush has a 60% chance of winning each match, then the probability that Ramsay will win the first three matches is $p\%$. Compute $100p$.

5. What is the largest number of equilateral triangles of side length 1 that can fit inside a regular hexagon of side length 3 without overlapping?

6. Working alone, Person A can fill an Olympic-sized swimming pool with concentrated molasses in 7000 years. Person B can do it in 5600 years, while Person C can do it in just 4000 years. How many years will it take for all three of them working together to fill the pool?

7. Alex owns a farm where he raises chickens and octopi. Each chicken has one brain and one heart, while each octopus has nine brains and three hearts. If Alex has a total of 168 brains and 96 hearts among his chickens and octopi, how many of those hearts are octopi hearts?

8. Alex is walking through a hallway. Every minute, there is a $\frac{1}{4}$ chance of a wild heff appearing, and a $\frac{1}{3}$ chance of a wild matthew appearing. (It is not possible for both to appear at the same time.) If the probability that Alex's first encounter is with a wild matthew and not a wild heff is expressed in reduced form as $\frac{m}{n}$, what is $100m + n$?

9. Let V be the volume of a regular octahedron with side length 2. Then if $V = \sqrt{\frac{m}{n}}$, where the fraction $\frac{m}{n}$ is in lowest terms, compute $100m + n$.

10. Jay's high school has a large number of students. He knows that 60% of them are boys, the other 40% of them are girls, and that 40% of the boys prefer pancakes over waffles. If overall, 40% of the students prefer waffles over pancakes (and no one is undecided) then what integer percentage of the girls prefer pancakes over waffles?

11. Isosceles triangle $\triangle ABC$ has $AB = AC = 7$ and $BC = 5$. Points D and E are chosen on sides AB and AC respectively so that $BD = DE = EC$. If the length of segment BD equals the reduced fraction $\frac{m}{n}$, then compute $100m + n$.

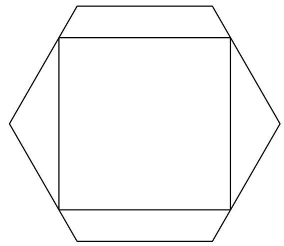
12. What is the greatest common divisor of 437 and 589?

13. A contest has 8 questions. Each question is written by Nir with probability $\frac{1}{3}$, and each of Nir's questions is about number theory with probability $\frac{1}{2}$. The remaining questions are written by Matthew, each of which is about number theory with probability $\frac{1}{4}$. (All probabilities are assumed to be independent.) The expected number of number theory questions on the contest is a reduced fraction $\frac{m}{n}$. Compute $100m + n$.

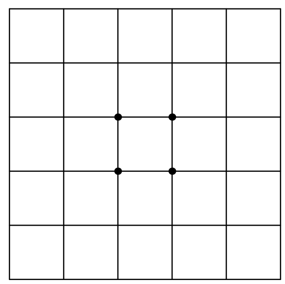
14. Theodore writes down all positive divisors of $10!$ divisible by 3. Eve then erases all of the even divisors. Compute the number of remaining divisors.

15. Madeline is climbing a 2000-meter tall mountain. She starts climbing from the base of the mountain at noon, and ascends at a constant speed of 10 meters per minute. After she has been climbing for some time, the wind begins blowing and slows her ascent to 6 meters per minute. If it is 4:00 PM on the same day when Madeline reaches the summit, how many minutes past noon was it when the wind started blowing?

16. In the diagram below, a square is inscribed in a regular hexagon of side length 1, and two of its opposite sides are parallel to two of the sides of the hexagon. The area of the square is $m - \sqrt{n}$, where m, n are positive integers. Compute $100m + n$.



17. How many paths, following the segments of the grid below and moving only up or right on each step, pass through one of the four vertices of the center square?



18. Let $ABCD$ be a unit square. Further, let P, Q, R, S be the midpoints of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$ respectively. Finally, let W, X, Y, Z be the midpoints of $\overline{AR}$, $\overline{BS}$, $\overline{CP}$, $\overline{DQ}$ respectively. If the area of quadrilateral $WXYZ$ is $\frac{m}{n}$ when expressed as a reduced fraction, find $100m + n$.

19. Alex starts with a bar of chocolate. Every second, he flips a fair coin, and if it comes up heads, he divides every chocolate bar in front of him into three smaller chocolate bars. (If it comes up tails, he does nothing.) Compute the expected number of chocolate bars he has after five seconds.

20. Call a positive integer *excellent* if the largest positive prime power that divides it is 9. For example, 18 is excellent because its divisors are 1, 2, 3, 6, 9, and 18; and out of these, 9 is the largest one that is a prime power. Compute the sum of all excellent positive integers.

21. In how many ways can six identical $3 \times 3 \times 1$ blocks be packed into a fixed $3 \times 3 \times 6$ box? (Count two ways as different if there is some block occupying some $3 \times 3 \times 1$ region of space in one arrangement, but no block occupying that exact same region of space in the other.)

22. What is the product of all positive integers $a < 10$ where $a! + 1$ is a perfect square?

23. Gina chooses four random positive one-digit integers a, b, c, and d, and she finds that $a \neq c$. The probability that the slope of the line connecting (a, b) and (c, d) is positive is written as a reduced fraction $\frac{m}{n}$. Compute $100m + n$.

24. Let b be a random integer from -7 to 3, and let a be a random integer from 1 to 3, inclusive. If the probability that the smaller solution for x in the quadratic equation $ax^2 + bx = a + b$ is 1 is the reduced fraction $\frac{m}{n}$, compute $100m + n$.

25. If a right triangle has perimeter 20 and area 16, the length of its hypotenuse is the reduced fraction $\frac{m}{n}$. Compute $100m + n$.

26. Yolanda is yelling positive integers. The first two numbers she yells are 1 and 3, and every following integer Yolanda yells is the product of the previous two. What is remainder when the 100th yelled number is divided by 32?

27. Points X and Y lie in the interior of regular hexagon $JUSTIN$ such that $IX = NX = UY = SY = XY = 1$. If $TX^2 = \frac{a+b\sqrt{c}}{d}$, where a, b, c, d are positive integers, c is not divisible by the square of a prime, and no prime divides all three of a, b, d, compute $1000a + 100b + 10c + d$.

28. Given that x and y are positive integers that satisfy $2x^2 + 8x = y^2 + 4y - 5$, what is the minimum value of $x + y$?

29. Anna throws a dart at a circular dartboard with radius 1, and Banana guesses a real distance between 0 and 1. Assuming Anna hits the dartboard randomly and Banana guesses randomly, the probability Anna's dart is closer to the center of the board than Banana's guess is the reduced fraction $\frac{m}{n}$. Compute $100m + n$.

30. Let A, B, C, D be points on a circle in that order such that AD is a diameter of the circle. Let E be a point on BD such that $ECD = 45°$. If $AD = 300$, $CD = 84$, and $AB = 180$, find DE.

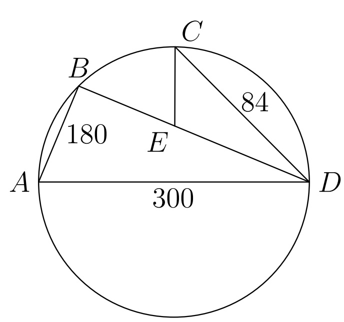


Diagram not to scale.

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Target Round
Problems 1 and 2

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This section of the competition consists of eight problems, which will be presented in pairs. Work on one pair of problems will be completed and answers will be collected before the next pair is distributed. The time limit for each pair of problems is six minutes. The first pair of problems is on the following page. When told to do so, go on to the next page and begin working. This round assumes the use of calculators, and calculations also may be done on scratch paper, but no other aids are allowed. All answers are integers, and units should not be included as part of your answer. Record only final answers in the Google form provided. If you complete the problems before time is called, use the remaining time to check your answers.

1. The FitnessGram PACER Test is a multistage aerobic capacity test that progressively gets more difficult as it continues. In particular, completing the first level requires runners to complete 7 laps, completing levels 2 and 3 requires runners to complete 8 laps each, completing levels 4 and 5 requires runners to complete 9 laps each, and the pattern continues with the number of laps increasing by one every two levels. How many total laps must be completed to complete the first 11 levels?

2. Let $ABCD$ be a square of side length 60, and let M and N be the midpoint of BC and CD respectively. Let the square $ABCD$ be folded along the lines AM, AN, and MN so that a tetrahedron is formed. Find the surface area of the tetrahedron.

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Target Round
Problems 3 and 4

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3. Pierce is doing a marathon play of a very difficult video game, which consists of a total of 26 levels. At the beginning, Pierce can complete two levels per day. However, due to fatigue and sleep deprivation, every day his rate of level completion will decrease by 10%. (Hence, for his second day he will only be able to complete 1.8 levels, 1.62 for the third, and so on.) What is the number of whole days it will take Pierce to complete half of the levels?

4. Eight congruent spheres are arranged so that each is externally tangent to three others, and the centers of the spheres are the vertices of a cube. Let there be two more spheres, one internally tangent to the original eight, one externally tangent. Suppose the ratio of the radius of the larger of these two spheres to the radius of the smaller is $m+\sqrt{n}$, where m and n are integers. Then what is $100m+n$?

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Target Round
Problems 5 and 6

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5. Eli has very interesting musical tastes. His playlist has N songs, three of which are by an obscure musician named Yobtu Ollaf, and the remaining $N-3$ of which are by other artists. When Eli shuffles his playlist and listens to all N songs on it in a random order, the probability that he hears all three Yobtu Ollaf songs in a row (regardless of what order they are in) is $\frac{1}{N}$. Compute N.

6. An isosceles trapezoid has its bases of length 2 and 8, and both its legs of length 5. Alex draws a circle inscribed in the trapezoid, meaning it lies inside the trapezoid and is tangent to all four of its sides. Olivier draws a circle circumscribed about the trapezoid, meaning it passes through all four of the trapezoid's vertices. If the ratio of the area of Olivier's circle to the area of Alex's circle is written in reduced form as $\frac{m}{n}$, compute $100m+n$.

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Target Round

Problems 7 and 8

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7. Compute the largest prime factor of $20! \times 21!$.

8. Call a sequence *nervous* if the quotient between each term and its previous is either $2, \frac{1}{2}$, or 3. Compute the number of nervous sequences made up of distinct positive integers with fewer than 4 digits, which start with 256 and end with 216.

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■ 2021 Austin Math Circle (Online) Practice Competition ■

Team Round

Problems 1 – 10

DO NOT BEGIN UNTIL YOU ARE INSTRUCTED TO DO SO

This section of the competition consists of 10 problems which the team has 20 minutes to complete. Team members may work together in any way to solve the problems. Team members may talk during this section of the competition. This round assumes the use of calculators, and calculations may also be done on scratch paper, but no other aids are allowed. All answers are integers, and units should not be included as part of your answer. The team captain must record the team's final answers in the Google form provided. None of the other team members should fill out the form.

1. Ultrasonic Josh is chasing a supersonic goose. The goose flies away from Josh at 600 m/s, and Josh runs at 3000 m/s. When a goose is flying directly away from him, it takes 10 seconds for Josh to catch it. How far did the goose travel in the time Josh was chasing it?

2. Jerry the ant drills into a $3 \times 4 \times 5$ rectangular prism made of unit cubes. He proceeds to drill from cube to adjacent cube, until he can no longer drill into an untouched cube. How many cubes can he drill into?

3. What is the sum of the coefficients of $(x - 2y + 3z)^{10}$ when expanded?

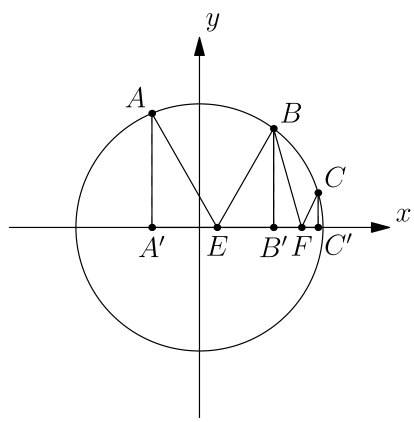
4. Compute the number of ordered pairs (a, b) of integer solutions to $a^2 - b^2 = 2021$.

5. Theo has less than 1000 followers on InstaPix. If you divide the number of followers he has by 7, 11, and 13, you get remainders of 4, 5, and 10, respectively. How many followers does Theo have?

6. Let A, B, and C be adjacent squares with side lengths 8, 12, and 8, respectively, each with two vertices on the horizontal line ℓ. Let O be the center of the circle passing through the 4 vertices among the three squares that do not touch the line ℓ. Let ℓ intersect the circle at points A and B. What is the value of AB^2?

7. Timmy plays a game with numbers, starting with the single number 1. Every turn, he flips a fair coin ($\frac{1}{2}$ probability of each result). If the coin flips heads, the number is multiplied by 3, otherwise, it is multiplied by $\frac{1}{3}$. If his expected score after 4 turns is $\frac{m}{n}$ when written as a reduced fraction, find $100m + n$.

8. Let A, B, C be points on circle ω in the coordinate plane centered at the origin such that B, C have positive x-coordinates, A does not, and the y-coordinates of A, B, C are $300, 260, 91$ respectively. Let A', B', C' be the feet of the altitudes from A, B, C to the x-axis. Let E and F be points inside the circle such that $AEA' \sim BEB'$ and $BFB' \sim CFC'$. If the radius r of the circle is 325, then $\frac{EF}{r} = \frac{m}{n}$, where the fraction $\frac{m}{n}$ is reduced. Find $100m + n$.



9. Leon is thinking of a quadratic polynomial $f(x)$ with integer coefficients with the property that $0 < f(1) < f(2) < f(0)$. He defines the function $g(x) = xf(x)f(f(x))$ and calculates $g(1) = 511$. Given this information, compute $g(3)$.

10. Andy is measuring the grand piano shown below, which consists of four quarter-circle arcs, each of which has rational radius, and two straight line segments. First, he finds that its perimeter is $20 + 6\pi$. Then, he holds his measuring stick in the unique way such that it touches the curved part of the piano twice, and finds that the distance between the two points where it touches is 8. If the area of the piano is $p + q\pi$ where p is an integer and q is rational, compute p.