Mock Mathcounts Problems & Solutions

Jay Leeds and Matthew Kroesche
Austin Math Circle

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Sprint Problems

Problem 1. What is the difference between four dozens and three half-dozens?

Solution. This difference is $48 - 18 = \boxed{30}$. □

Problem 2. What is the sum of the two-digit numbers that are multiples of five but not of two or three?

Solution. There aren't very many of these, so we can just list them out. $25+35+55+65+85+95 = \boxed{360}$. □

Problem 3. At a store, a certain coat costs \$100 in November. In December, the store puts the coat on sale for 25% off. When the sale ends in January, the discount price is marked up by 30%. What is the price of the coat at the end of January?

Solution. A decrease by 25% is the same as multiplication by 0.75. Similarly, an increase by 30% is the same as multiplication by 1.3. So, our answer is $\$100 \cdot 0.75 \cdot 1.3 = \boxed{\$97.50}$. □

Problem 4. A *Metro News One* forecast predicts a 50% chance of rain on Monday and a 40% chance of rain on Tuesday. Find the probability that it will rain on Monday or Tuesday (or both). Express your answer as a percentage.

Solution. The probability that it won't rain on Monday is 50%, while the probability that it won't rain on Tuesday is 60%, so the probability it won't rain on either day is 30%. Thus, the probability it will rain at least once is $\boxed{70\%}$. □

Problem 5. Maggie scored an average of 88 points on five quizzes. What score would she need to earn on her sixth quiz to raise her average to 90?

Solution. If x is the score she needs, we have $\frac{x+440}{6} = 90$, so $x + 440 = 540$, so $x = \boxed{100}$. □

Problem 6. The seven points A, B, C, D, E, F, and G are evenly spaced in alphabetical order on segment AG, which has length 210. Find the length of segment CF.

Solution. Since AG consists of six smaller line segments of equal length, each of those segments has length 35. CF contains three of these segments, so its length is $\boxed{105}$. □

Problem 7. Barbara earned the following scores on eight math tests: $94, 86, 97, 95, 92, 85, 93$, and 99. What is the median of Barbara's test scores? Express your answer as a decimal to the nearest tenth.

Solution. In order, Barbara's scores are $85, 86, 92, 93, 94, 95, 96$, and 97. The median of a list with an even number of elements is the average of the two innermost elements, which is $\frac{93+94}{2} = \boxed{93.5}$. □

Problem 8. In a technology club, every student likes math or science. 19 of the students like math, 22 students like science, and 7 like both. How many students are in the club?

Solution. There are $19 + 22 = 41$ students between the two subjects, but we've double-counted the seven who like both math and science, so we subtract them out for a total of $\boxed{34}$ students. □

Problem 9. A certain math test has fifteen questions. The answer to each question is a positive integer. All of the questions except question #8 have the same answer, which is an even integer, and the sum of the answers to all fifteen questions is 586. What is the smallest possible value of the answer to question #8?

Solution. Let the common answer be x and the answer that is different be y. Then $14x + y = 586$, and since x is even, $14x$ is a multiple of 28. The largest multiple of 28 that is less than 586 is 560, so the least possible value of y is $\boxed{26}$, which corresponds to $x = 40$.
Proposed by Matthew Kroesche. □

Problem 10. What is the largest number that can appear in a list of five unique integers with median 5 and range 5?

Solution. If the range is 5, the best we can do with these restrictions is to find the greatest possible minimum value and to add five to that (to get the greatest value). Since the median, or the third entry, is 5, the second entry is at most one less, or 4, and the first entry is thus at most 3, so the greatest possible number in the list is $\boxed{8}$. □

Problem 11. Jim writes the expression $1 + 2 + 3 + 4 + 5$ on a blackboard. Then, Dwight erases two of the addition symbols and replaces them with other standard arithmetic operations: $-, *$, or $\div$. After that, Michael inserts a single set of parentheses into the equation. What is the smallest possible value of the new expression? For example, one possible final expression is $1 - 2 + (3 + 4 \div 5)$.

Solution. We seek to minimize the value of this addition by adding one pair of parentheses and changing two of the sum operations to other operations. Clearly, we will want to subtract out a large value, so we change the first $+$ to a $-$ and attach parentheses around 2 through 5, giving $1 - (2 + 3 + 4 + 5)$. Then, we want to grow the value in the parentheses as much as possible, so we shift one of the parentheses and multiply the largest number by the whole sum to get $1 - (2 + 3 + 4) * 5 = \boxed{-44}$. □

Problem 12. Let $f^1(x) = x^2 - 1$, and let $f^n(x) = f^1(f^{n-1}(x))$ for $n > 1$. Find $f^{100}(1)$.

Solution. $f^1(1) = 0$, $f^2(1) = f^1(0) = -1$, $f^3(1) = f^1(f^2(1)) = f^1(-1) = 0$, so we observe a cycle: Odd values of f^n will be 0 and evens will be -1. Hence, $f^{100}(1) = \boxed{\text{-1}}$. □

Problem 13. A wooden cube of side length one is cut into two pieces by a plane that is parallel to one of its faces. What is the sum of the surface areas of the two pieces?

Solution. This equals the total surface area of the cube, which is 6, plus the area of the two new faces produced by the cut, each of which is a unit square and thus has area 1. So the total surface area is $6 + 1 + 1 = \boxed{8}$.
Proposed by Matthew Kroesche. □

Problem 14. Chord AB of length 20 is drawn in a circle with center O and radius 26. Find the distance from O to the midpoint of AB.

Solution. Let the midpoint of AB be C. $AC = BC = 10$ and $\angle ACO$ is right, so $CO^2 = AO^2 - AC^2 = 676 - 100 = 576$, giving $CO = \boxed{24}$. □

Problem 15. Two standard fair six-sided dice are rolled. What is the probability that the sum of the numbers on top of the two dice is prime? Express your answer as a common fraction.

Solution. There are 36 possible results in total. We count the results that will get us a prime sum. There is one way (rolling two ones) to roll two. Two (two, then one and one, then two) to roll three, four (four and one, three and two, and the reverses) to roll five, six to roll seven, and two to roll eleven. This is a total of $1 + 2 + 4 + 6 + 2 = 15$ ways, so our answer is $\frac{15}{36} = \boxed{\dfrac{5}{12}}$. □

Problem 16. Find the positive real number x that satisfies $2^{x^2} = 4^{x^4}$. Express your answer as a common fraction in simplest radical form.

Solution. We know $2^{x^2} = 2^{2x^4}$, so $x^2 = 2x^4$. Since $x > 0$, $x^2 = \frac{1}{2}$ and $x = \boxed{\dfrac{\sqrt{2}}{2}}$.
Proposed by Matthew Kroesche. □

Problem 17. Find the sum of the squares of the values of x satisfying $x^3 - ax^2 + bx - c = 0$. Express your answer in terms of a, b, and c.

Solution. Let the roots be x_1, x_2, and x_3. Note that by Vieta's Formulas, $a = x_1 + x_2 + x_3$, so $a^2 = x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3$. Similarly, $b = x_1x_2 + x_1x_3 + x_2x_3$, so $2b = 2x_1x_2 + 2x_1x_3 + 2x_2x_3$. Thus, $\boxed{a^2 - 2b} = x_1^2 + x_2^2 + x_3^2$. (Because of the unusual form of the problem, a wide variety of equivalent answers will be accepted.) $\square$

Problem 18. Let point A be $(0, 0)$, and let point B be $(6, 0)$. Point C is randomly selected on the line segment connecting $(0, 3)$ and $(6, 3)$. What is the expected value of the area of triangle ABC?

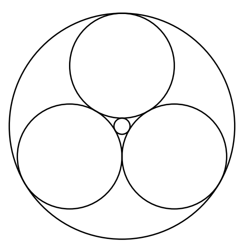
Solution. The area of a triangle is determined by its base and its height, and since the perpendicular from C to AB has constant length, the area of ABC will actually always be the same! The base is $AB = 6$, the height is the distance between $y = 3$ and $y = 0$, which is 3, and the area is thus $\boxed{9}$. $\square$

Problem 19. For how many positive integers n at most 100 does the decimal representation of $\frac{1}{n}$ terminate?

Solution. A fraction terminates if its denominator has no prime factors other than 2 and 5. Let's count everything up based on the number of powers of five... With no powers of five, we have $1, 2, 4, 8, 16, 32, 64$. That's seven. With one power of five, we have $5, 10, 20, 40, 80$. That's five. With two powers of five, we have $25, 50, 100$. That's three.

So, the total count is $\boxed{15}$. $\square$

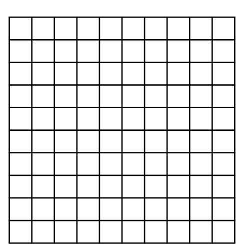
Problem 20. Three circles of radius one are drawn on a sheet of paper such that each is externally tangent to the other two. Chris draws two more circles on the paper. One of these circles is externally tangent to the former three circles while the other is internally tangent to the circles. What is the product of the areas of these two new circles? Express your answer as a common fraction in terms of π.



Solution. Connecting the centers of the three unit circles gives us an equilateral triangle of side length 2, and the distance from a vertex of the triangle to its center is $\frac{2\sqrt{3}}{3}$. Since this is also the center of the two circles that Chris draws, the radii of these two circles are $r = \frac{2\sqrt{3}}{3} - 1$ and $R = \frac{2\sqrt{3}}{3} + 1$, so $Rr = \frac{4}{3} - 1 = \frac{1}{3}$. Then $(\pi R^2) \cdot (\pi r^2) = \pi^2 \cdot (Rr)^2 = \boxed{\dfrac{\pi^2}{9}}$.

Proposed by Matthew Kroesche. $\square$

Problem 21. How many rectangles of any size have their sides contained in the edges of the ten-by-ten of unit squares shown below?



Solution. Each rectangle is uniquely determined by choosing 2 of the 11 horizontal lines and 2 of the 11 vertical lines, and letting the rectangle be the area enclosed by these four lines. Thus the answer is $\binom{11}{2}^2 = 55^2 = \boxed{3025}$.

Proposed by Matthew Kroesche. $\square$

Problem 22. What is the hundreds digit of 11^{2018}?

Solution. $11^{2018} = (10+1)^{2018} = \binom{2018}{0}10^0 + \binom{2018}{1}10^1 + \binom{2018}{2}10^2 + \cdots + \binom{2018}{2018}10^{2018}$. Since we want the hundreds digit, we only need to worry about the first three terms, which evaluate to $1 + 2018 \cdot 10 + 1009 \cdot 2017 \cdot 100$. We see that the units digit of $1009 \cdot 2017$ is 3, and adding to that the tens digit of 2018 (since we multiplied it by 10^1) we get $\boxed{4}$ as our answer.
Proposed by Matthew Kroesche. □

Problem 23. A car has a faulty odometer with a display that resets back to zero immediately upon reaching a certain number of miles. At first, the odometer reads 215 miles. Then, after the car is driven for a while, the odometer has reset once and finally reads 422 miles. When the car is driven the same length again, the odometer resets twice more and eventually reads 56 miles. At what number of miles does the odometer reset?

Solution. Suppose the car resets at x miles. Then, the difference between $x + 422$ and 215 is equal to the difference between $3x + 56$ and $x + 422$. Thus, $x + 207 = 2x - 366$, so $x = \boxed{573}$. □

Problem 24. The permutations of the letters in the word CIRCLE are listed in alphabetical order. In what position will the word CLERIC be listed? For example, the permutation $CCEILR$ appears in position one while $CCEIRL$ appears in position two.

Solution. C is the first letter in alphabetical order, and there are 120 permutations of IRCLE, so CLERIC will be in the first 120 permutations. L is fourth in alphabetical order among the letters in IRCLE, so CLERIC will be somewhere between permutations 73 and 96 (as it is in the fourth quintile of the 120 permutations). Then, E is the second letter in alphabetical order in the "word" IRCE, so CLERIC will be between permutations 79 and 84, the second quartile. R is the last letter in alphabetical order in IRC, so CLERIC will be in position 83 or 84. Since R comes after C, CLERIC is in position $\boxed{84}$. □

Problem 25. Let p be the probability that when a coin is flipped n times, no two consecutive flips show heads. Find the value of n that minimizes $|p - \frac{1}{3}|$.

Solution. We do constructive counting, simulating the process of throwing the coin. After a tails flip, the next flip can be heads or tails, but after a heads flip, the next flip has to be tails. This lets us construct the following table:

n	Tails	Heads	Total
1	1	1	2
2	2	1	3
3	3	2	5
4	5	3	8
5	8	5	13
6	13	8	21
7	21	13	34
8	34	21	55
9	55	34	89
10	89	55	144

A clever student may notice that these values correspond to the Fibonacci sequence, allowing us to compute the table far more quickly.

We see that for any n, there are 2^n outcomes in total. We observe that for $n = 5$, $p = \frac{13}{32} = \frac{26}{64}$, for $n = 6$, $p = \frac{21}{64}$, and for $n = 7$, $p = \frac{17}{64}$. Clearly, $n = \boxed{6}$ gives the closest value to $\frac{1}{3}$. □

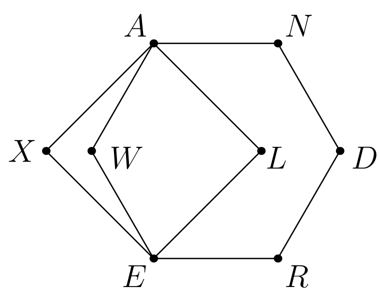
Problem 26. Compute $1^2 + 3^2 + 5^2 + \cdots + 49^2$.

Solution. Since $1^2+2^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$, we have $1^2+3^2+\cdots+49^2 = (1^2+2^2+\cdots+50^2)-(2^2+4^2+\cdots+50^2) = (1^2+2^2+\cdots+24^2)-4(1^2+2^2+\cdots+12^2) = \frac{50\cdot51\cdot101}{6} - \frac{4\cdot25\cdot26\cdot51}{6} = \frac{50\cdot51\cdot(101-52)}{6} = \frac{600\cdot23}{6} = \boxed{20825}$.
Proposed by Matthew Kroesche. □

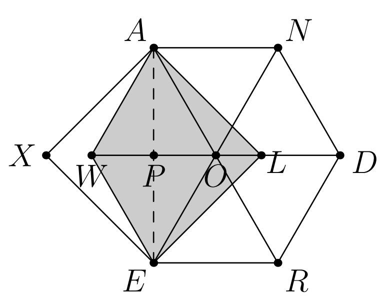
Problem 27. Albert and Betty each pick a digit (from zero to nine), but they don't tell each other which digit they picked. When their numbers are multiplied together, the result's final digit is x. Albert says, "As far as I can tell, x could be any of the ten digits." Betty responds, "I wasn't sure before, but now I know what x is." Find x, assuming Albert and Betty have both used correct logic.

Solution. Albert's information tells us that based on his number, the product of the two values could end in any of the ten digits. This tells us that Albert's number is not even (because that would guarantee an even product) and not five (for similar reasons), so it is $1, 3, 7$, or 9. The fact that Betty didn't know what the product would be before implies that her number was not zero, because if it were zero she would know that the last digit of the product would be zero. The only other number for which she would know x based on this new information is 5, since five times an odd always ends in $\boxed{5}$. ☐

Problem 28. Regular hexagon $ANDREW$ and square $ALEX$ lie in the same plane, as shown in the figure. What fraction of the area of hexagon $ANDREW$ is also inside of square $ALEX$? Express your answer as a common fraction in simplest radical form.



Solution. Partition the hexagon $ANDREW$ into six equilateral triangles each having a vertex at its center O, and without loss of generality let them all have side length 1. Then the area of the hexagon is $6 \cdot \frac{\sqrt{3}}{4} = \frac{3\sqrt{3}}{2}$. Now consider triangle AWL. It has the same altitude as equilateral AWO, which can be shown to be $\frac{\sqrt{3}}{2}$ by dividing AWO into two 30-60-90 right triangles. Since $AWL \cong EWL$, $AE = 2 \cdot AP = \sqrt{3}$, where AP is the foot of the altitude from A to WL. Finally, since P is the midpoint of AE, it is the center of square $ALEX$ and so $LP = \frac{\sqrt{3}}{2}$ as well, and $WP = \frac{1}{2}$. Thus $LW = \frac{1+\sqrt{3}}{2}$, the area of AWL is $\frac{3+\sqrt{3}}{8}$, and the area of $ALEW$ is twice that, or $\frac{3+\sqrt{3}}{4}$. Then the fraction of hexagon $ANDREW$ that overlaps with $ALEX$ is $\frac{\frac{3+\sqrt{3}}{4}}{\frac{3\sqrt{3}}{2}} = \boxed{\frac{1+\sqrt{3}}{6}}$.



Proposed by Matthew Kroesche. ☐

Problem 29. Paul chooses an integer r uniformly at random from the set $\{0, 1, 2, 3, 4\}$. He then places r red balls and $4 - r$ blue balls into an urn. Amelia draws a ball from the urn, sees that it is red, and places it back in the urn. Afterwards, Claire draws a ball from the urn, sees that it also is red, and places it back in the urn. Now, Susan is about to draw a ball from the urn. What is the probability that it will be red? Express your answer as a common fraction.

Solution. The probability that a red ball is drawn is $\frac{r}{4}$, and so the probability that two red balls are drawn with replacement is $\frac{r^2}{16}$. Since r is chosen at random, the probability of this event is $\frac{1}{5}(\frac{0+1+4+9+16}{16}) = \frac{3}{8}$. Furthermore, the conditional probability that a third red ball will be drawn given that the first two balls were also red is equal to the probability that all three balls will be red, divided by the probability that the first two will be red. But the probability of three red balls in a row is $\frac{r^3}{64}$, and again since r varies, the expected probability would be $\frac{1}{5}(\frac{0+1+8+27+64}{64}) = \frac{5}{16}$. Then the final answer is simply $\frac{\frac{5}{16}}{\frac{3}{8}} = \boxed{\frac{5}{6}}$.
Proposed by Matthew Kroesche. □

Problem 30. The number 104060405 is the product of four distinct prime numbers, all of which are less than 10000. What is the largest of these four primes?

Solution. Dividing out a 5, we see that $104060405 = 5 \times 20812081 = 5 \times 10001 \times 2081$. Since all four primes that divide the number are given to be less than 10000, 10001 must be the product of two such primes, and 2081 must be prime. Furthermore, since 10001 is not divisible by 2, 3, or 5, its two prime factors are both less than 2081 so the answer is $\boxed{2081}$. (10001 turns out to factor as 73×137.)
Proposed by Matthew Kroesche. □

Problem 1. Joe is counting by threes. The first number he says is 100, and the second number he says is 103. Find the hundredth number he says.

Solution. If Joe starts with 100, his number will increase by three ninety-nine times to get to the tenth number, so it will increase by $3 \cdot 99 = 297$. Thus, he will say $\boxed{397}$ hundredth. (Alternatively, let s_n be the nth number he says. We can quickly see that $s_n = 3n + 97$, which quickly gives our answer.) □

Problem 2. A truck starts with 15 gallons of fuel in its tank and drives 50 miles. Then, 10 more gallons of fuel are put into the tank and the truck drives 42 miles. Finally, 18 more gallons of fuel are added and the truck drives another 60 miles. What is the minimum fuel efficiency, in miles per gallon, that would allow the truck to complete this process without ever running out of fuel? Express your answer as a decimal to the nearest hundredth.

Solution. To make it through the first drive, the truck's fuel efficiency must be at least $\frac{50}{15} \approx 3.33$. To get through the second drive, the efficiency must be at least $\frac{92}{25} = 3.68$. To make it through the third drive, it needs at least $\frac{152}{43} \approx 3.53$ miles per gallon. The most restrictive of these bounds is $\boxed{3.68}$ miles per gallon. □

Problem 3. A certain company runs a two-digit number of factories that each produce a greater two-digit number of widgets per day. In total, the company produces 1547 widgets every day. Find the number of factories the company runs.

Solution. Factoring, we find $1547 = 7 \cdot 13 \cdot 17$. We then list the ways to express 1547 as the product of two numbers and find that there is only one way to express it as the product of two two-digit numbers: $17 \cdot 91$. Therefore, there are $\boxed{17}$ factories each producing 91 widgets per day. □

Problem 4. Richard has an unfair coin that comes up heads with probability p. If he flips the coin six times, the probability that exactly three of the flips are heads is 0.02. Find the largest possible value of p. Express your answer as a common fraction in simplest radical form.

Solution. The probability of getting exactly three heads when flipping the coin six times is $\binom{6}{3}p^3(1-p)^3$, which we know equals $\frac{1}{50}$. Thus, since $\binom{6}{3} = 20$, we have that $p^3(1-p)^3 = \frac{1}{1000}$ and so $p(1-p) = \frac{1}{10}$. Then $10p^2 - 10p + 1 = 0$ and so $p = \frac{10 \pm \sqrt{10^2 - 4 \cdot 10 \cdot 1}}{2 \cdot 10}$. We want the larger of these two roots, which is $\boxed{\dfrac{5+\sqrt{15}}{10}}$.

Proposed by Matthew Kroesche. □

Problem 5. Find the area of the figure in the Cartesian coordinate plane bounded by the circle with center $(0,0)$ and radius 4 and above the lines $y = 0$ and $y = x$. Express your answer in terms of π.

Solution. Drawing the figures, we see that line $y = 0$ will take out the lower half of the circle while $y = x$ will remove another $45°$ arc. Thus, we have $135°$ left, and since the area of the circle is 16π, the area of this figure is $\frac{135}{360} \cdot 16\pi = \boxed{6\pi}$. □

Problem 6. Eight teams are competing in a single elimination basketball tournament. Every team is assigned a unique integer rank from one to eight. In every round of the tournament, each team is randomly assigned to an opponent and plays a match against them. The winning team advances to the next round and the losing team is eliminated. In every game, the team with the numerically lower rank is the winner. Find the expected value of the sum of the ranks of the teams that play against the team with rank one. Express your answer as a common fraction.

Solution. In the first round, clearly, all other ranks are equally likely to appear as the opponent, so the expected value for this round is $\frac{2+3+\cdots+8}{7} = 5$.

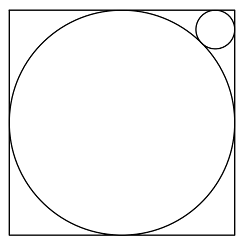
In the second round, there is a $\frac{6}{7}$ chance rank two is still in the tournament (as there is a one-seventh chance it was matched up with rank one), a $\frac{5}{7}$ chance rank three is still in the tournament (similar logic), and so on. Thus, the expected value in the second round is $\frac{2\cdot\frac{6}{7}+3\cdot 57+\ldots+7\cdot\frac{1}{7}}{\frac{6}{7}+\frac{5}{7}+\ldots+\frac{1}{7}} = \frac{11}{3}$.

Finally, in the third round, there is a $\frac{4}{7}$ chance rank two is still alive in the tournament, a $\frac{2}{7}$ chance rank three is still alive in the tournament, a $\frac{4}{35}$ chance rank four is still alive in the tournament, and a $\frac{1}{35}$ chance rank five is still alive in the tournament. Thus, the expected value for this round is $2\cdot\frac{4}{7}+3\cdot\frac{2}{7}+4\cdot\frac{4}{35}+5\cdot\frac{1}{35} = \frac{13}{5}$.

So, the expected value of the sum, by linearity of expectations, is $5+\frac{11}{3}+\frac{13}{5} = \boxed{\frac{169}{15}}$.

□

Problem 7. A circle is inscribed in a square, and another circle is externally tangent to the first circle and tangent to two sides of the square, as shown in the figure below. What is the ratio of the area of the large circle to the area of the small circle? Express your answer in simplest radical form.



Solution. Without loss of generality, let the big circle have radius 1, so that the square has side length 2. Then the distance from the center of the big circle to any vertex of the square is $\sqrt{2}$, and the distance from the center of the small circle to the top right vertex of the square is $r\sqrt{2}$, where r is the radius of the small circle. But then we have $1 + r + r\sqrt{2} = \sqrt{2} \implies r = \frac{\sqrt{2}-1}{\sqrt{2}+1} = 3 - 2\sqrt{2}$, and then $r^2 = 17 - 12\sqrt{2}$. Finally, the ratio of the big circle to the small circle is $\frac{\pi}{\pi(17-12\sqrt{2})} = \boxed{17+12\sqrt{2}}$.

Proposed by Matthew Kroesche. □

Problem 8. Tomas writes two three-digit positive base-ten integers (without leading zeros), one of which is three times the other, on a blackboard. Later, Ben sees the blackboard, but he assumes the two integers on it are written in base 16. Under this assumption, he converts the two integers to base 10 and divides the larger one by the smaller one. What is the difference between the largest and smallest possible values of the rational number Ben wrote? Express your answer as a common fraction.

Solution. Let the smaller of the two integers be $\underline{A}\,\underline{B}\,\underline{C}_{10}$. Write out the multiplication $\underline{A}\,\underline{B}\,\underline{C} \times 3$, and let X and Y be the digits that are carried in the tens and hundreds places, respectively. Note that $X, Y \in \{0,1,2\}$ and $A \in \{1,2,3\}$. Now the larger of the two is $\underline{D}\,\underline{E}\,\underline{F}_{10}$, where $F = 3C - 10X$, $E = 3B + X - 10Y$, and $D = 3A + Y$. (As a check, $100D + 10E + F = 300A + 30B + 3C$.) Now we simply divide. The fraction that Ben writes is equal to $\frac{256D+16E+F}{256A+16B+C} = \frac{256(3A+Y)+16(3B+X-10Y)+(3C-10X)}{256A+16B+C} = \frac{3(256A+16B+C)+96Y+6X}{256A+16B+C} = 3+\frac{6(X+16Y)}{256A+16B+C}$. Now it is clear that the smallest possible value is 3, attained when $X = Y = 0$. Meanwhile, we can intuitively maximize this when $X = Y = 2$, in which case the numerator is 204. The denominator in this case must have C at least 7 and B at least 6, and then we take $A = 1$ for a minimum of $167_{16} = 359$. Quickly testing other values of X and Y shows us that the numerator shrinks much more quickly than the denominator. For example $X = Y = 1$ results in a denominator of at least $134_{16} = 308$ – however, the numerator is cut in half to 102, which clearly gives a smaller fraction. Thus the difference between the largest and smallest possible values is simply $(3+\frac{204}{359}) - 3 = \boxed{\frac{204}{359}}$.

Proposed by Matthew Kroesche. □

Problem 1. A perfect number is a number that is equal to the sum of its factors, not including itself. The third-smallest perfect number is 496. How many factors does it have (including itself)?

Solution. $496 = 2^4 * 31$, hence, by a well-known formula, it has $(4+1)(1+1) = \boxed{10}$ factors. □

Problem 2. Ted takes out a \$1000 loan at a local bank. The first year, it accumulates 6% interest. After two years, Ted owes \$1113. Find the interest rate Ted paid during the second year. Assume that Ted does not pay off any of his debt and that interest is compounded (e.g. the first year's interest accumulates interest during the second year). Express your answer as a percentage.

Solution. At a 6% interest rate, Ted will owe $\$1000 \cdot 1.06 = \$1060$ after the first year. Thus, in the second year, his debt is multiplied by $\frac{1113}{1060} = 1.05$, so Ted pays $\boxed{5\%}$ interest during the second year. □

Problem 3. A rectangular tablet has a diagonal of length seven inches, while a similar rectangular tablet has a diagonal of length ten inches. What is the ratio of the area of the second tablet to that of the first tablet? Express your answer as a common fraction. Two rectangles are similar if their side-lengths are in the same ratio.

Solution. Let one rectangle have lengths a and ka and the other have lengths b and kb. Then, $\sqrt{a^2 + k^2a^2} = 7$ while $\sqrt{b^2 + k^2b^2} = 10$. We want to find $\frac{b^2}{a^2}$, so we square these equations and divide to get $\frac{b^2+k^2b^2}{a^2+k^2a^2} = \frac{100}{49}$, which is equivalent to $\frac{b^2(k^2+1)}{a^2(k^2+1)} = \frac{100}{49}$, so $\frac{b^2}{a^2} = \boxed{\frac{100}{49}}$. □

Problem 4. Eric randomly picks a positive integer n from 1 to 1000. What is the expected value of the number of digits in the decimal representation of n? Express your answer as a decimal to the nearest thousandth.

Solution. There are 9 1-digit numbers, 90 2-digit numbers, 900 3-digit numbers, and 1 4-digit number in the range from 1 to 1000. Thus the expected number of digits is $\frac{9\cdot1+90\cdot2+900\cdot3+1\cdot4}{1000} = \frac{9+180+2700+4}{1000} = \frac{2893}{1000} = \boxed{2.893}$ digits.
Proposed by Matthew Kroesche. □

Problem 5. A frog hops around the perimeter of a rectangular lake from one corner to the opposite corner. At the same time, a toad, starting at the same corner as the frog, swims directly across the lake to the opposite corner. The frog and the toad reach the opposite corner of the lake at the same time. If the speed at which the frog hops is 40% faster than the speed at which the toad swims, what is the ratio of the length of the longer side of the lake to the length of its shorter side? Express your answer as a common fraction.

Solution. If the frog moves 40% faster than the toad, then his speed is 1.4 times the toad's speed, and thus the distance he travels must be 1.4 times the distance the toad travels since they both take the same amount of time. Now we let r be the ratio of the longer side of the lake to the shorter side, and a be the length of the shorter side. Then the distance the frog travels is $a + ar = a(r+1)$, and the distance the toad travels is $\sqrt{a^2 + (ar)^2} = a\sqrt{r^2+1}$. Thus $r + 1 = 1.4\sqrt{r^2+1}$. Multiplying by 5 so as to simplify the arithmetic, $5r + 5 = 7\sqrt{r^2+1}$, and then squaring both sides gives $25r^2 + 50r + 25 = 49r^2 + 49$. Finally, we subtract and divide by 2 to obtain $12r^2 - 25r + 12 = 0$. This factors as $(3r-4)(4r-3) = 0$, and so $r \in \{\frac{3}{4}, \frac{4}{3}\}$. Since r is the ratio of the longer side to the shorter side, $r = \boxed{\frac{4}{3}}$.
Proposed by Matthew Kroesche. □

Problem 6. A spider is located at one vertex of a rectangular prism with sides of lengths 1, 2, and 3, and a fly is situated on the vertex furthest from the spider. What is the length of the shortest path over the surface of the prism that the spider can take to catch the fly? Express your answer in simplest radical form.

Solution. If we "unfold" the prism, we see that there are three different straight-line paths the spider could take depending on which two faces it crawls across. The lengths of these three paths are $\sqrt{1^2+(2+3)^2} = \sqrt{26}$, $\sqrt{2^2+(1+3)^2} = \sqrt{20} = 2\sqrt{5}$, and $\sqrt{3^2+(1+2)^2} = \sqrt{18} = 3\sqrt{2}$. The smallest of these three is $\boxed{3\sqrt{2}}$.
Proposed by Matthew Kroesche. □

Problem 7. What is the least positive integer n such that the number $\dfrac{9009}{n}$ does not terminate with a finite number of digits when written as a decimal?

Solution. We factor $9009 = 3^2 \cdot 7 \cdot 11 \cdot 13$. Now any prime other than 2 or 5 that divides n to a higher power than it divides 9009 will result in a repeating decimal. (There are no restrictions on how many times the primes 2 and 5 can divide n since they are also factors of the base.) Thus the first prime that is not allowed to divide n is 17, and since we can confirm that no number less than 17 is divisible more than once by 7, 11, 13 or more than twice by 3, our answer is indeed $\boxed{17}$.
Proposed by Matthew Kroesche. □

Problem 8. Wolstenholme and Zsigmondy compete head-to-head to determine which of the two is a better mathematician. They are given a single number theory problem to solve, and both solve it in a two-digit integer number of seconds. Afterwards, they observe that

- The product of their times is a perfect square
- The sum of their times is a prime number
- When Wolstenholme's time w and Zsigmondy's time z are concatenated, the result $100w + z$ is a perfect square

How many seconds passed before the first person solved the problem?

Solution. Since $w + z$ is prime, $\gcd(w, z) = 1$. Otherwise $\gcd(w, z) = a > 1$ and $w = ar$, $z = as$ with r, s positive and then $w + z = ar + as = a(r + s)$, where $a, r + s > 1$ and thus $w + z$ is composite, a contradiction. Now, since wz is a perfect square and w, z are relatively prime, w and z must both be perfect squares themselves, since they can have no prime factors in common. Finally, note that $100w$ is a perfect square. For $100w + z$ to be a perfect square as well (recall $z < 100$) it must be true that $\sqrt{100w} < 50 \implies 100w < 2500 \implies w < 25$. But then it follows that $w = 16$, and then $z = 81$. Thus Wolstenholme solves the problem first, in a mere $\boxed{16}$ seconds. □

Problem 9. A "time-number" is a six-digit number $\overline{abcdef}$ (i.e. with units digit f, tens digit e, etc) such that $\overline{ab}$ is an integer, potentially written with leading zeroes, between zero and twenty-three and $\overline{cd}$ and $\overline{ef}$ are both integers, possibly written with leading zeroes, between zero and fifty-nine. For example, 123456, 235959, and 000000 are time-numbers, but 999999 is not. How many time-numbers $\overline{abcdef}$ exist such that $\overline{abcdef}$ is a palindrome but $\overline{abcd}$ is not?

Solution. The six-digit number is a palindrome when $\overline{ab} = \overline{fe}$, which is possible only when b is between 0 and 5. Thus, there are 16 possible values of $\overline{ab}$ that lead to reasonable values of $\overline{ef}$. Then, $\overline{cd}$ must itself be a palindrome, and thus there are six possible values: $00, 11, 22, 33, 44$, and 55. Thus, $16 \cdot 6 = 96$ such numbers are six-digit palindromes. However, if $\overline{cd} = 00, 11$, or 22, $\overline{ab}$ and $\overline{ef}$ can be the same number, resulting in a valid palindrome in both $abcd$ and $abcdef$ forms. Therefore, we must subtract these three cases, for a total of $\boxed{93}$ valid numbers. □

Problem 10. An operation "$\clubsuit$" maps every pair of positive real numbers x, y to a positive real number $x \clubsuit y$. The following identities hold for this operation for all positive real numbers x, y:

$$x \clubsuit y = y \clubsuit x$$
$$x \clubsuit x = \frac{x}{2}$$
$$2x \clubsuit 2y = 2(x \clubsuit y)$$
$$\frac{x \clubsuit (y+1)}{y \clubsuit (x+1)} = \frac{x(y+1)}{y(x+1)}$$

Find the value of $20 \clubsuit 17$. Express your answer as a common fraction.

Solution. By the second identity, $20 \clubsuit 17 = \frac{1}{2}(40 \clubsuit 34)$. Now, we replace x with $x-1$, and rearrange the third identity taking advantage of the commutative property:

$$x \clubsuit y = \frac{xy}{(x-1)(y+1)} \cdot ((x-1) \clubsuit (y+1))$$

From this identity we can establish the following:

$$40 \clubsuit 34 = \frac{40 \cdot 34}{39 \cdot 35} \cdot (39 \clubsuit 35)$$

$$39 \clubsuit 35 = \frac{39 \cdot 35}{38 \cdot 36} \cdot (38 \clubsuit 36)$$

$$38 \clubsuit 36 = \frac{38 \cdot 36}{37 \cdot 37} \cdot (37 \clubsuit 37)$$

And from the first identity, $37 \clubsuit 37 = \frac{37}{2}$. Thus:

$$20 \clubsuit 17 = \frac{1}{2}(40 \clubsuit 34) = \frac{1}{2} \cdot \frac{40 \cdot 34}{39 \cdot 35} \cdot \frac{39 \cdot 35}{38 \cdot 36} \cdot \frac{38 \cdot 36}{37 \cdot 37} \cdot \frac{37}{2} = \frac{1}{2} \cdot \frac{40 \cdot 34}{37 \cdot 37} \cdot \frac{37}{2} = \boxed{\frac{340}{37}}.$$

(As a side note, the explicit form of the operation turns out to be $x \clubsuit y = \frac{xy}{x+y}$.)
Proposed by Matthew Kroesche. ☐

Problem 1. Let $f(x) = 10x + 7$. Compute $f(f(f(f(f(1)))))$.

Solution. This function appends a "7" to the base 10 representation of its input. Thus, applying it five times successively to 1 will give results of 17, 177, 1777, 17777, and $\boxed{177777}$.
Proposed by Matthew Kroesche. □

Problem 2. Janet buys a bag of jellybeans. She eats 30% of them, loses 8 of them, eats 20% of what remains, shares 24 with her friends, eats 70% of the remaining ones, and gives the last 6 to her younger brother. How many jellybeans were in the bag when she bought it?

Solution. We work backwards. She has six before giving the last few away, $6 \cdot \frac{10}{3} = 20$ before she eats for the third time, 44 before she shares some with her friends, 55 before eating for the second time, 63 before losing some, and thus $\boxed{90}$ before eating for the first time. □

Problem 3. How many elements do the sequences $4, 7, 10, \cdots, 100$ and $4, 8, 12, \cdots, 100$ share?

Solution. These values are the multiples of the four and the numbers that leave a remainder of one when divided by three. Therefore, their intersection will be the sequence $4, 16, \cdots, 100$. This sequence has the same amount of terms as the sequence $12, 24, \cdots, 108$ (as we have simply added eight to every element), which has the same number of terms as $1, 2, \cdots, 9$ (as we have divided by twelve), so the sequence has $\boxed{9}$ elements. □

Problem 4. What is the least prime number that can be written as the sum of three distinct prime numbers?

Solution. If one of these three primes is 2, then the other two are odd, and so the sum is even and cannot be prime. If we don't use 2, the next three smallest primes we can use are 3, 5, and 7, but $3 + 5 + 7 = 15$, which is not prime. Finally, we replace 7 with the next larger prime, 11, to get $3 + 5 + 11 = \boxed{19}$, which is our answer.
Proposed by Matthew Kroesche. □

Problem 5. A hare jumps at a rate of 100 units per second, but after every ten seconds of jumping it takes a five-second break to cool off. A tortoise crawls along the same path at a rate of 80 units per second without breaks. After how many seconds will the tortoise first be 1500 units ahead of the hare? Express your answer as a decimal to the nearest hundredth.

Solution. Every fifteen seconds, the tortoise advances $80 \cdot 15 - 100 \cdot 10 = 200$ units ahead of the hare. So, in $15 \cdot 8 = 120$ seconds, the tortoise will be 1600 units ahead of the hare. But the tortoise only needs to be 1500 units ahead of the hare, so he can stop $\frac{100}{80} = 1.25$ seconds earlier, after $\boxed{118.75}$ seconds. □

Problem 6. Find the integer n that satisfies $17n^4 + 29n^3 + 42n^2 + 31n + 1 = 0$.

Solution. By the Rational Root Theorem, the only possible integer solutions are $n = 1$ and $n = -1$. Since $n = 1$ gives an obviously positive value, the answer must be $\boxed{-1}$. Sure enough, plugging in $n = -1$ gives $17 - 29 + 42 - 31 + 1 = 0$. □

Problem 7. The sum of the area and the perimeter of a rectangle is 64. The lengths of the rectangle's sides are each increased by two. Find the new area of the rectangle.

Solution. Let the side-lengths of the rectangle be x and y. Then, the area is xy and the perimeter is $2x+2y$. Thus, $xy + 2x + 2y = 64$. We're hoping to find the new area, which is $(x + 2)(y + 2) = xy + 2x + 2y + 4$, so we add four to our equation to find that $(x + 2)(y + 2) = xy + 2x + 2y + 4 = \boxed{68}$. □

Problem 8. Triangle ABC is similar to triangle XYZ. Given $\angle B = 80^\circ$, $AB = YZ$, and $BC = XY$, find $\angle Z$.

Solution. Since $AB = YZ$ and $BC = XY$, $\frac{AB}{XY} = \frac{YZ}{BC}$. But from our similarity, $\frac{AB}{XY} = \frac{BC}{YZ}$, so $\frac{BC}{YZ} = \frac{YZ}{BC}$, so $\frac{BC}{YZ}^2 = 1$, so $BC = YZ$, so the ratio of our similarity is one. Thus, $XY = BC = YZ$, so XYZ is isosceles with vertex Y. Thus, since $\angle Y = \angle B = 80^\circ$, $\angle Z = \boxed{50^\circ}$. ☐

Problem 9. Find the difference between the largest and smallest possible areas of a right triangle with two sides of lengths 21 and 29. Express your answer as a common fraction.

Solution. For the largest area, 21 and 29 will be legs, giving an area of $\frac{609}{2}$. For the smallest area, 29 will be the hypotenuse, so the other leg will be $\sqrt{29^2 - 21^2} = 20$ units long. Thus, the area will be 210. The difference is $\frac{609}{2} - 210 = \frac{189}{2}$. ☐

Problem 10. If $x\%$ of 1000 equals 10% of kx, find k.

Solution. $x\%$ of 1000 equals $10x$ while 10% of kx equals $0.1kx$, so $10x = 0.1kx$ and $k = \boxed{100}$. ☐

Problem 11. Josh converts the base-ten integers 1 through 729 into base three and writes the results on a blackboard. How many times does he write the digit 1?

Solution. Consider the parallel situation in which Josh writes all of the six-digit base three umbers, possibly with leading zeros, on the blackboard. This will include the integers zero through 728, since 729 is seven digits. Clearly, by symmetry, exactly $\frac{1}{3}$ of the digits in the six-digit base three numbers will be 1. With 729 numbers and six digits, there are $\frac{729 \cdot 6}{3} = 1458$ ones.

Josh will actually write these same numbers, but he will not write zero (which will not change the number of ones) and will write 729 (which is 1000000_3, so this will add one zero). Thus, the total is $\boxed{1459}$ ones. ☐

Problem 12. Jonathan and Jackson each pick an integer from one to ten independently and at random. Find the probability that Jonathan's integer is greater than Jackson's. Express your answer as a common fraction.

Solution. There is a $\frac{1}{10}$ chance that the two integers will be the same. If they are the same, Jonathan's will obviously not be greater, but otherwise, there is a $\frac{1}{2}$ chance his will be larger, so our answer is $\frac{9}{10} \cdot \frac{1}{2} = \boxed{\frac{9}{20}}$.
Proposed by Matthew Kroesche. ☐

Problem 13. Joy has five thin rods, one each of every integer length from one to five. Find the number of distinct obtuse triangles Joy can create by using three of these rods as the sides.

Solution. A triangle is obtuse if the sum of the squares of its two smaller lengths is less than the square of its largest length. However, the triangle inequality mandates that the sum of these lengths (without squaring) is greater than the largest length.

Let's do casework on the smallest side-length. It is impossible to form any triangles using the rod of length one, as the smallest possible difference between the two larger lengths is one, which would violate the triangle inequality.

With the rod of length two, we can form a triangle with the rods of lengths three and four or four and five. Both are obtuse, because $2^2 + 3^2 < 4^2$ and $2^2 + 4^2 < 5^2$.

With the rod of length three, we can only form a three-four-five right triangle, which clearly is not obtuse.

Therefore, there are $\boxed{2}$ possible obtuse triangles to be formed. ☐

Problem 14. Find the radius of the circle inscribed in a triangle with sides of lengths $13, 14,$ and 15.

Solution. By Heron's Formula, the area of this triangle is $\sqrt{21 \cdot 8 \cdot 7 \cdot 6} = 84$. From $A = rs$ (area equals inradius times semiperimeter), $21r = 84$, so $r = \boxed{4}$. ☐

Problem 15. In the year 2100, Martians trade in two currencies: Zigs and zags. If two zigs and three zags are with \$131 and three zigs and two zags are worth \$219, how much would it cost to buy one zig and one zag?

Solution. Let a zig cost x dollars and a zag cost y dollars. Then, $2x + 3y = 131$ and $3x + 2y = 219$. Adding these equations, $5x + 5y = 350$, so $x + y = \boxed{\$70}$. □

Problem 16. John begins to read a book on Monday. He reads ten pages on Monday, and every day after that, he reads ten pages more than twice the number of pages he read the day before. How many pages of the book will he read on the next Friday?

Solution. This problem can be solved via simple computations. On Monday, he reads ten pages, so on Tuesday, he reads thirty, on Wednesday, seventy, on Thursday, 150, and on Friday, $\boxed{310}$. □

Problem 17. The sum of the infinite geometric series $1 + x + x^2 + x^3 + x^4 + ...$ is n. Find x in terms of n. Express your answer as a common fraction in terms of n.

Solution. It is well-known that the sum of an infinite geometric series with first term one and common ratio x is $\frac{1}{1-x}$, so $\frac{1}{1-x} = n$. Then, $n - nx = 1$, so $nx = n - 1$, so $x = \boxed{\frac{n-1}{n}}$. □

Problem 18. Evaluate

$$\frac{1}{\sqrt{1}+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \cdots + \frac{1}{\sqrt{2017}+\sqrt{2018}}.$$

Express your answer in simplest radical form.

Solution. Note that $\frac{1}{\sqrt{n+1}+\sqrt{n}} = \frac{\sqrt{n+1}-\sqrt{n}}{(\sqrt{n+1}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})} = \sqrt{n+1} - \sqrt{n}$. Then the entire sum telescopes: $(\sqrt{2}-\sqrt{1}) + (\sqrt{3}-\sqrt{2}) + (\sqrt{4}-\sqrt{3}) + \cdots + (\sqrt{2018}-\sqrt{2017}) = \boxed{\sqrt{2018}-1}$.
Proposed by Matthew Kroesche. □

Problem 19. Evaluate $\dfrac{2+4+6+\cdots+2018}{1+3+5+\cdots+2017}$. Express your answer as a common fraction.

Solution. This equals $\frac{1009\cdot 1010}{1009^2} = \boxed{\dfrac{1010}{1009}}$, due to the formulas for the sum of consecutive even numbers and consecutive odd numbers.
Proposed by Matthew Kroesche. □

Problem 20. A number is a *sqube* if it is the product of a square and a cube of two distinct prime integers. How many positive integers less than or equal to 1000 are squbes?

Solution. We do casework on the number being cubed.

If 2 is being cubed, then the other number must be a square at most 125. This means that it can be at most 11, since $11^2 = 121$, so $3, 5, 7$, and 11 can be squared.

If 3 is being cubed, the number being squared must be less than or equal to $\lfloor \frac{1000}{27} \rfloor = 37$, so 2 or 5 can be the number being squared.

If 5 is being cubed, the square must be at most 8, so 2 must be the number being squared.

If 7 is being cubed, the square must be less than 3, so there are no possible numbers to be squared. Obviously, any larger primes will fail to work for the same reason.

Thus, there are $\boxed{7}$ squbes less than or equal to 1000. □

Problem 21. If $a \diamondsuit b = ab - 2a - 2b + 907$, what is $1 \diamondsuit (2 \diamondsuit (3 \diamondsuit (4 \diamondsuit 5)))$?

Solution. Let $N = 907$. Note that $2 \diamondsuit b = 2b - 4 - 2b + N = N - 4$ independent of b. Thus $1 \diamondsuit (2 \diamondsuit (3 \diamondsuit (4 \diamondsuit 5))) = 1 \diamondsuit (N-4) = 1 \cdot (N-4) - 2 \cdot 1 - 2 \cdot (N-4) + N = N - 4 - 2 - 2N + 8 + N = \boxed{2}$, independent of N.
Proposed by Matthew Kroesche. □

Problem 22. Find the maximum area of a circle drawn within a 8 in. by 6 in. rectangle.

Solution. The largest such circle has radius 3, as any larger circle would not fit given the shorter dimension, so the largest possible area is $\boxed{9\pi}$. $\square$

Problem 23. A four-digit palindrome is chosen at random. Find the probability that it is divisible by eleven. Express your answer as a percentage.

Solution. A four-digit palindrome is of the form $abba$ for digits a and b. It is well-known that a number is congruent to the alternating sum of its digits modulo eleven. Therefore, $abba = a - b + b - a = 0(mod 11)$. Thus, no matter the selection of a and b, the palindrome will be divisible by 11, giving an answer of $\boxed{100\%}$. $\square$

Problem 24. A pyramid has a square base of length 4. Its other four faces are equilateral triangles. Find the volume of the pyramid. Express your answer as a common fraction in simplest radical form.

Solution. Clearly, the base area is 16. The height of one of the triangular faces is $4 \cdot \frac{\sqrt{3}}{2} = 2\sqrt{3}$. The right triangle containing the foot of the altitude of the pyramid, the foot of the altitude of the triangle, and the vertex of the pyramid has one leg equal to the height, one leg equal to $\frac{4}{2} = 2$, and hypotenuse equal to $2\sqrt{3}$, so the height of the pyramid is $2\sqrt{2}$. Since in a pyramid, $V = \frac{bh}{3}$, the volume is equal to $16 \cdot 2\sqrt{2} \cdot \frac{1}{3} = \boxed{\frac{32\sqrt{2}}{3}}$. $\square$

Problem 25. Twenty teams are playing in a double elimination volleyball tournament. In a double elimination tournament, teams play games of volleyball against each other until all but one of the teams have been eliminated. When a team loses two games, it is eliminated and leaves the tournament. Given that no ties occur, find the sum of the maximum and minimum possible numbers of games that could be played in the tournament.

Solution. With 20 teams, 19 teams must be eliminated, losing two games, and one must have either zero losses or one loss. For 19 teams to lose two games each, 38 games must be played, the minimum number of games. If the last team has lost a game, one more game must be played, for a total of 39. Therefore, the answer is $38 + 39 = \boxed{77}$. $\square$

Problem 26. One day, Jan solves a single math problem. The next day, she solves three problems. Each following day, she solves three times as many problems as the day before. How many math problems will she have solved in a week?

Solution. One could just do the computations... $1 + 3 + 9 + 27 + 81 + 243 + 729 = \boxed{1093}$.
But there is a better way. This is a geometric series, and as a result, we can express $1+3+3^2+3^3+3^4+3^5+3^6$ as $\frac{3^7-1}{3-1} = \boxed{1093}$. $\square$

Problem 27. A date is *gargantuan* if the month, day, and last two digits of the year, in that order, form a geometric progression whose common ratio is neither an integer nor the reciprocal of an integer. For example, the last gargantuan date was September 12, 2016, since $(9, 12, 16)$ is a geometric progression with common ratio $\frac{4}{3}$. What is the next gargantuan date?

Solution. If the common ratio is $\frac{a}{b}$ in lowest terms, with $a, b > 1$, then b^2 must divide the month and a^2 must divide the year. If there is a gargantuan date in 2018, the only value of a that would work is 3 since $3^2 | 18$. Then the day would be $6b$ and the month would be $2b^2$, and thus we must have $b = 2$ since the month is at most 12. Then we obtain $(8, 12, 18)$ as our geometric sequence. This is indeed a valid date, so the answer is $\boxed{\text{August 12, 2018}}$. *Proposed by Matthew Kroesche* $\square$

Problem 28. Find $2^5 + 2^6 + 2^7 + 2^8 + 2^9 + 2^{10}$.

Solution. Obviously, one could just calculate these values and sum them up. But, to save time, note that the sum $2^0 + 2^1 + 2^2 + ... + 2^n = 2^{n+1} - 1$ for any n. Applying this to $n = 4$, $2^0 + 2^1 + 2^2 + 2^3 + 2^4 = 2^5 - 1$. Adding this to our original equation shows that our desired value is the sum for $n = 10$, minus $2^5 - 1$ (since that's the value we added). For $n = 10$, the sum is $2^{11} - 1$, so we subtract $2^5 - 1$ to get a result of $2^{11} - 2^5 = \boxed{2016}$. □

Problem 29. A right triangle has one leg of length 10. Additionally, its hypotenuse is 2 units longer than its other leg. Find the length of this second leg.

Solution. We let the second leg be x. We have $x^2 + 100 = (x+2)^2$ by the Pythagorean Theorem. This gives $x^2 + 100 = x^2 + 4x + 4$, or $4x + 4 = 100$. This gives $x = \boxed{24}$. □

Problem 30. In a certain class, 8 students received a grade of 90% on an exam while the rest earned 100%. If the average grade in the class is 96%, how many students were in the class?

Solution. If x students earn 100%, we have $\frac{100x+720}{x+8} = 96$, or $100x + 720 = 96x + 768$. This gives $4x = 48$, so $x = 12$. Thus, there are $12 + 8 = \boxed{20}$ students in the class. □

Problem 31. Find the product $\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{3}\right)\left(1 + \frac{1}{4}\right)\cdots\left(1 + \frac{1}{2018}\right)$. Express your answer as a common fraction.

Solution. We are asked to multiply $\frac{3}{2}$, $\frac{4}{3}$, $\frac{5}{4}$, and so on. Multiplying the first few values, we get $\frac{3}{2} \cdot \frac{4}{3} = \frac{4}{2} = 2$. Then, $2 \cdot \frac{5}{4} = \frac{5}{2}$. We observe a pattern: The product is the final numerator divided by two. (This can easily be proved with induction, though in MATHCOUNTS it is usually sufficient to simply find the pattern without proof). This presents an answer of $\boxed{\dfrac{2019}{2}}$. □

Problem 32. What is the distance between two opposite vertices of a cube with side-length 3? Express your answer in simplest radical form.

Solution. The distance is $\sqrt{3^2 + 3^2 + 3^2} = \boxed{3\sqrt{3}}$. □

Problem 33. Della has some coins worth a total of \$1.87, exactly n of which are pennies. For how many values of n is this possible?

Solution. Since all the denominations of coins except for pennies are evenly divisible by 5, the number of pennies Della has must be 2 more than a multiple of 5 so that her remaining coins have total value that is exactly divisible by 5. (In this case, we can achieve the result if all of these coins are nickels.) So the question reduces to the number of integers between 0 and 187 (inclusive) that are 2 more than a multiple of 5. These are $5 \cdot 0 + 2, 5 \cdot 1 + 2, \cdots, 5 \cdot 37 + 2 = 187$, so there are a total of $\boxed{38}$.
Proposed by Matthew Kroesche. □

Problem 34. Solve: $2^4 + 2^4 + 2^4 + 2^4 = 4^x$.

Solution. $2^4 \cdot 4 = 2^6 = 4^3$, so $x = \boxed{3}$. □

Problem 35. Equilateral triangle TRI and regular hexagon $HEXAGO$ have the same area. What is the ratio of the side length of TRI to that of $HEXAGO$? Express your answer in simplest radical form.

Solution. Let TRI have side length a and $HEXAGO$ have side length b. Then $\frac{a^2\sqrt{3}}{4} = \frac{6b^2\sqrt{3}}{4} \implies a^2 = 6b^2 \implies \frac{a}{b} = \boxed{\sqrt{6}}$.
Proposed by Matthew Kroesche. □

Problem 36. A math team consists of three boys and one girl. What is the probability that a randomly chosen pair of students from the team consists of two boys? Express your answer as a common fraction.

Solution. This equals $\frac{\binom{3}{2}}{\binom{4}{2}} = \frac{3}{6} = \boxed{\frac{1}{2}}$.
Proposed by Matthew Kroesche. □

Problem 37. Find the sum of the values of x for which the value of the fraction $\frac{1}{2+\frac{3}{x+4}}$ is undefined. Express your answer as a decimal to the nearest tenth.

Solution. A fraction is undefined if its denominator is zero. So, if $x + 4 = 0$, which occurs when $x = -4$, the fraction will be undefined. Similarly, if $2 + \frac{3}{x+4} = 0$, the fraction will be undefined. Multiplying by $x + 4$, this is equivalent to $2x + 11 = 0$, or $x = -5.5$. Thus, our answer is $-4 - 5.5 = \boxed{-9.5}$. □

Problem 38. Lucas rolls a fair eight-sided die and Dustin rolls a fair twelve-sided die. What is the expected value of the sum of the two die rolls?

Solution. The expected value of an n-sided die is $\frac{1+2+\cdots+n}{n} = \frac{n(n+1)}{2n} = \frac{n+1}{2}$, and by linearity of expectations the expected value of the sum is equal to the sum of the expected values, which is $\frac{8+1}{2} + \frac{12+1}{2} = \boxed{11}$.
Proposed by Matthew Kroesche. □

Problem 39. Solve for x: $4^{x+1} - 2^{x+2} + 1^{x+4} = 0$.

Solution. Let $2^x = y$. The equation becomes with some manipulation:

$$4y^2 - 4y + 1 = 0.$$

Factoring shows this equation to be equal to

$$(2y - 1)^2 = 0.$$

Thus, we see that $y = \frac{1}{2}$. Solving $2^x = \frac{1}{2}$, we find $x = \boxed{-1}$. □

Problem 40. How many perfect powers are less than one thousand? (A perfect power is defined as x^y for any positive integers x and y with $y > 1$.)

Solution. $32^2 = 1024$, so there are 31 squares less than $1,000$. Similarly, there are 9 cubes, 3 fifth powers, and 2 seventh powers. The rest of the powers (fourth, sixth, etc) will already have been counted in these categories, so that's all of them. However, 3 of the cubes are squares ($1^3, 4^3$, and 9^3), so there are 6 new cubes. Similarly, one of the fifth powers and one of the seventh powers are overcounted (since 1 was already included in our squares), so our answer is $31 + 6 + 2 + 1 = \boxed{40}$. □

Problem 41. A number is *explosive* if it is divisible by five or contains the digit five in its base-ten representation. For example, $5, 505$, and 653 are all explosive, but $1, 14$, and 113 are not. How many positive integers less than or equal to $1,000$ are *explosive*?

Solution. For a number to be divisible by five or to contain the digit five, it must satisfy one or more of three conditions: First, that it has a five as the hundreds digit, second, that it has a five as the tens digit, and third, that it has a five or a zero as the units digit. We use the Principle of Inclusion and Exclusion to solve for the number that have one or more. This principle gives that the desired value is the sum of the amounts of numbers that meet each condition minus the sum of the amounts of numbers that meet each pair of conditions plus the amount of numbers that meet all three conditions.

Because we have one choice for the hundreds digit and ten each for the other two digits, there are 100 numbers that meet the first condition. Similarly, 100 numbers meet the second condition. Since we have two choices for the ones digit and ten for both others, there are 200 numbers that meet the third question.

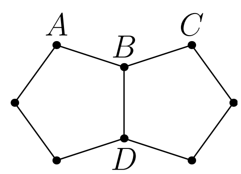
With one choice each for the hundreds and tens digits and ten for the last, there are 10 numbers that meet the first and second conditions. With one choice for the hundreds digit, two for the ones, and ten for the tens, there are 20 numbers that meet the first and third conditions. Similarly, there are 20 numbers that meet the last two conditions.

Finally, with one choice each for the hundreds and tens digits and two for the units digit, 2 numbers meet all three conditions. Thus, there are $100 + 100 + 200 - 10 - 20 - 20 + 2 = \boxed{352}$ *explosive* numbers up to 1000. □

Problem 42. A positive integer factor of $6! = 720$ is randomly chosen. What is the probability that it is prime? Express your answer as a common fraction.

Solution. We have that $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 2^4 \cdot 3^2 \cdot 5$, which has $5 \cdot 3 \cdot 2 = 30$ positive integer factors, three of which are prime. Thus the answer is $\frac{3}{30} = \boxed{\frac{1}{10}}$.
Proposed by Matthew Kroesche. ☐

Problem 43. Two regular pentagons share side BD, as shown below. A third regular polygon is drawn with AB and BC as two of its sides. How many sides does this polygon have?



Solution. The interior angles of a regular n-gon each measure $180(1 - \frac{2}{n})^\circ$. Plugging in $n = 5$ gives the interior angles of each of the pentagons as $\angle ABD = \angle CBD = 108^\circ$. Thus the interior angle of the third polygon is $\angle ABC = 360 - 2 \cdot 108 = 144^\circ$, and thus $144 = 180(1 - \frac{2}{n}) \implies 1 - \frac{2}{n} = \frac{4}{5} \implies \frac{2}{n} = \frac{1}{5}$, and so $n = \boxed{10}$.
Proposed by Matthew Kroesche. ☐

Problem 44. How many ways are there to place three distinguishable tokens in a three by three grid so that every row and every column contain exactly one token?

Solution. There are nine ways to place the first token. Then, there are four places in the grid where the second token can go (as it must be in one of the other two rows and columns), and then one place for the final token. Thus, there are $9 \cdot 4 = \boxed{36}$ configurations. ☐

Problem 45. Simplify $(2 + 0)^{(1+8)} - (2 + 0)^{(1+7)}$.

Solution. This equals $2^9 - 2^8 = 2^8 = \boxed{256}$.
Proposed by Matthew Kroesche. ☐

Problem 46. Suppose $a^1 + a^{-1} = 2$. Find $a^4 + a^3 + a^2 + a^1 + a^0 + a^{-1} + a^{-2} + a^{-3} + a^{-4}$.

Solution. Squaring the first equation and subtracting two, $a^2 + a^{-2} = 2$. Similarly, $a^4 + a^{-4} = 2$. Multiplying the second equation by the first, $a^3 + a + a^{-1} + a^{-3} = 4$, so $a^3 + a^{-3} = 2$, hence our desired sum is $2 + 2 + 2 + 2 + a^0 = 2 + 2 + 2 + 2 + 1 = \boxed{9}$. Alternatively, note that $a = 1$ (or solve for a to find the same result), which trivially gives the same sum. ☐

Problem 47. If $4x + 2 = 8$, find $8x + 2$.

Solution. Multiplying the given equation by two, we have $8x + 4 = 16$, so $8x + 2 = \boxed{14}$. ☐

Problem 48. Let a, b, c be positive real numbers such that $\frac{bc}{a} = \frac{1}{2}$, $\frac{ca}{b} = 8$, and $\frac{ab}{c} = 128$. What is the greatest possible value of $a + b + c$?

Solution. Multiplying pairs of equations together, we find that $a^2 = 1024$, $b^2 = 64$, and $c^2 = 4$. Thus $|a| = 32$, $|b| = 8$, and $|c| = 2$. Clearly the maximum value of $a + b + c$ is when a, b, c are all positive, which is $32 + 8 + 2 = \boxed{42}$. *Proposed by Matthew Kroesche.* ☐

Problem 49. Let a, b, and c denote the number of distinguishable arrangements of the letters in the words $AUSTIN$, $MATH$, and $CIRCLE$ respectively. Evaluate $a+b+c$.

Solution. We have $a = 6! = 720$, $b = 4! = 24$, and $c = \frac{6!}{2!} = 360$, for a total of $720 + 24 + 360 = \boxed{1104}$.
Proposed by Matthew Kroesche. □

Problem 50. Compute $\dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{6} - \dfrac{1}{24}$. Express your answer as a common fraction.

Solution. This equals $\frac{12+6+4-1}{24} = \frac{21}{24} = \boxed{\dfrac{7}{8}}$.
Proposed by Matthew Kroesche. □

Problem 51. The number $9999\cdots9$, with a total of 2016 nines, is divisible by seven. Find the sum of its three smallest prime factors.

Solution. Two and five are obviously not factors. Since the number is divisible by 99, it is divisible by three and eleven (we can write the number as $99(10^0 + 10^2 + 10^4 + \cdots + 10^{2014})$. This means that the sum of the smallest prime factors is $3 + 7 + 11 = \boxed{21}$. □

Problem 52. Express $0.\overline{123}$ as a common fraction.

Solution. Let $x = 0.\overline{123}$. Then, $1000x = 123.\overline{123}$, so $999x = 123$, so $x = \frac{123}{999} = \boxed{\dfrac{41}{333}}$. □

Problem 53. Evaluate $\frac{1}{2}^{(-2)^2 \cdot \frac{-1}{2}}$.

Solution. The value in the exponents equals $\frac{-1}{2} \cdot 4 = -2$, and $\frac{1}{2}^{-2} = \boxed{4}$. □

Problem 54. Find the sum of the values of x that satisfy $\frac{101}{x} = \frac{x}{37}$.

Solution. Expressed as a quadratic, this equation is equivalent to $x^2 - 3737 = 0$, which, by Vieta's Formulas, has roots with sum $\boxed{0}$. □

Problem 55. A new flag design consists of a rectangle with all four corners connected by straight lines to a point X in the flag's interior. Two opposite corners have distances 8 and 5 to X, while a third corner is 7 units away from X. Find the distance from the final corner to the interior point. Express your answer in simplest radical form.

Solution. Let the flag be rectangle $ABCD$ with interior point X. Then, let the distance from A to the foot from X to AB be x_1, the distance from B to this foot be x_2. Also, let the distance from B to the foot from X to BC be y_1, and the distance from C to this foot be y_2.

WLOG, let $AX = 8$, $BX = 7$, and $CX = 5$. Then, the Pythagorean Theorem gives $x_1^2 + y_1^2 = 64$, $x_2^2 + y_1^2 = 49$, and $x_2^2 + y_2^2 = 25$. Then, subtracting the second equation from the sum of the first two gives $x_1^2 + y_1^2 = 40$. Since the desired length DX is $\sqrt{x_1^2 + y_1^2}$, it is equal to $\sqrt{40} = \boxed{2\sqrt{10}}$.

This fact can actually be generalized: $AX^2 + CX^2 = BX^2 + DX^2$. This identity is known as the British Flag Theorem. □

Problem 56. In a certain country, license plates consist of a sheet of gray metal containing three vowels (a, e, i, o, or u) followed by five digits. Then, the rules are changed so that "y" is considered to be a vowel (in addition to the original five). In addition, the country begins to allow ten colors of license plates, including the original gray. How many more distinct license plates can be formed under the new system than under the old one?

Solution. Under the new system, $6^3 \cdot 10 = 2160$ vowel-color-schemes can be created (six ways to choose each vowel and ten for the color), as opposed to $5^3 = 125$ under the old one. Thus, there are 2035 new schemes. Each can have 10^5 sets of digits to go with them, so this is a total of $203,500,000$ new plates. □

Problem 57. The difference between two perfect squares is 133. Given that their square roots have difference greater than one, find the sum of these squares.

Solution. Let the squares be x^2 and y^2. We have $x^2 - y^2 = 133$, so $(x+y)(x-y) = 133$. $133 = 7 \cdot 19$, and since $x - y$ is less than $x + y$, $x - y$ is either one or seven. Since we are given that $x - y > 1$, $x - y = 7$, so $x + y = 19$, so $x = 13$ and $y = 6$. This gives $x^2 + y^2 = \boxed{205}$. $\square$

Problem 58. The roots of the polynomial $x^2 - 41x + 378$ are the numbers of diagonals in a regular n-gon and a regular m-gon. Find $n + m$.

Solution. An n-gon has $\frac{n(n-3)}{2}$ diagonals. Since 41, the sum of the roots of this polynomial, is fairly small, we can test some possible values of n and see if they give us a value of m. For small n, the numbers of diagonals are $2, 5, 9, 14, 20, 27, 40$, and after that they become too big. Using guess-and-check, we can see that 14 and 27 are the roots of this polynomial. Hence $n, m = 7, 9$, so $n + m = \boxed{16}$. $\square$

Problem 59. A deck of cards contains 52 cards, four each of 13 different ranks. A set of five cards is a *full house* if three of the cards are of the same rank and the other two cards are of the same rank (distinct from the first set of three). How many *full houses* can be formed using the cards from a standard deck?

Solution. There are 13 ways to pick the first rank, 12 ways to pick the second, 4 ways to pick three cards from the first rank, and 6 ways to pick two cards from the second, a total of $\boxed{3744}$ hands. $\square$

Problem 60. Nevin glues 27 identical 6-sided dice together to form a $3 \times 3 \times 3$ cube. Each die is oriented randomly and independently of the others. Given that any two opposite sides on a die show values summing to seven, find the expected number of prime numbers visible on the surface of the cube.

Solution. The three primes from 1 to 6 are 2, 3, and 5, and their orientation on the die, which has 2 and 5 opposite each other and 3 on a different face, is symmetric. Now, for every possible orientation of the dice, if there are n primes visible on the surface of the cube, there is an "opposite" orientation where every die is rotated 180 degrees about the axis through the 1 and 6 faces, and then 90 degrees about the axis through the 3 and 4 faces. This will result in every prime being replaced by a non-prime and vice versa, and thus the number of primes visible on the surface will be the total number of faces visible on the surface, which is $6 \cdot 3 \cdot 3 = 54$, minus n, and together these two orientations will have an average of 27 primes visible on their surface. But this operation as described is reversible, so we can pair off all possible orientations in this way, and thus the expected value itself is $\boxed{27}$ prime numbers.
Proposed by Matthew Kroesche. $\square$

Problem T. How many permutations a_1, a_2, a_3, a_4, a_5 of the positive integers one through five exist such that for any non-empty proper subset of the permutation, the sum of the values of the a_i in the subset is not equal to the sum of the values of i in the subset?

For example, the sequence $3, 2, 1, 5, 4$ does not satisfy this condition, because $a_4 + a_5 = 4 + 5$. However, $5, 1, 2, 3, 4$ is satisfactory.

If your answer is a and the correct answer is c, your score for this problem will be the greater of $\frac{a}{c}$ and $\frac{c}{a}$. Whichever student has the lower score will receive the better rank. If this criterion is insufficient to break all ties, the student to submit an answer first will win.

Solution. Observe that the permutation can fail to meet these conditions based on a subset of one, two, three, or four elements, but if the condition fails on a subset of three of the elements, the subset of the other two elements will also fail, because if the sum of the three elements is x, the sum of the other two is $1 + 2 + 3 + 4 + 5 - x = 15 - x$, and likewise for the sum of the values of i. Similar logic shows that if the condition fails on a subset of four elements, it will also fail for the one remaining element. So, we consider only the potential failures for subsets of one or two elements.

First, we compute how many permutations will not fail for a subset of one element. Clearly, to meet this condition, we need only that $a_i \neq i$ for all i.

We first pick a_1, which can be any of the four remaining values. WLOG, let $a_1 = 2$. Then, we choose a_2. There are two cases from here: $a_2 = 1$ and $a_2 \in 3, 4, 5$.

Case 1: $a_2 = 1$: In this case, we must find the permutations a_3, a_4, and a_5 among $3, 4, 5$. Since there are only six possibilities, we just list them out: $(a_3, a_4, a_5) \in (3, 4, 5), (3, 5, 4), (4, 3, 5), (5, 4, 3)$ will fail, but $(a_3, a_4, a_5) = (4, 5, 3)$ or $(5, 3, 4)$ will work. So, we have $4 \cdot 2 = 8$ cases here.

Case 2: $a_2 \in 3, 4, 5$. WLOG, assume $a_2 = 3$ (we actually have three choices for this, so we will multiply by three afterwards). If $a_3 = 1$, there is clearly only one way to arrange a_4 and a_5, so we have 12 cases. If $a_3 = 4$ or 5, WLOG, let a_3 equal four (we similarly actually have two choices here). Then, a_4 must equal five (since a_5 can't equal five), so we have 24 more cases. This is a total of 36 extra cases.

So, we have 44 permutations that satisfy the first condition.

Now, we remove the permutations that satisfy the first condition but not the second.

We break into cases. Let the bad subset be a_i, a_j.

Case 1: $a_i = j, a_j = i$. In this case, there are ten possible pairs i, j, and there are two ways to arrange the other three elements in a way that satisfies condition one (see Case 1 above), so this gives 20 total permutations.

Case 2: All other situations. We see that there are just a few ways for $a_i + a_j = i + j$ if $a_i \neq j$ and $a_j \neq i$. In particular, we could have $1 + 4 = 2 + 3$, $1 + 5 = 2 + 4$, or $2 + 5 = 3 + 4$. This gives us six possible pairs i, j. WLOG, suppose $i, j = 1, 4$. Then, $a_1 = 2$ and $a_4 = 3$, or vise versa. We must now arrange $1, 4, 5$ among a_2, a_3, and a_5. There are four ways to do this (as we assign a_5 to either 1 or 4 and can arrange the other two however we'd like), giving us a total of $6 \cdot 2 \cdot 4 = 48$ permutations.

Now, we add back the permutations we double-counted.

Observe that there are no permutations that have two case 1 failures. This is because if $a_i = j$, there is only one value of j that can create a bad permutation. Thus, the two pairs must be disjoint. But then, the permutation does not satisfy our first condition as if, for example, our pairs are $1, 2$ and $3, 4$, $a_5 = 5$.

Now, we seek situations in which there are two case 2 failures. This is impossible if the pairs are disjoint. However, consider the case when we choose two non-disjoint pairs from case 2. There are six ways to do pick two pairs so that the values they must be equal to will intersect (which is necessary to create two simultaneous failures). Suppose those pairs are $1, 4$ and $1, 5$: In this case, $a_1 = 2$, $a_4 = 3$, and $a_5 = 4$ works, so a_2 and a_3 are one and five, giving us two ways to arrange them. This means there are 12 ways to proceed in this case.

Now, consider situations in which there is a case 1 failure and a case 2 failure. Consider our example case two failure with $a_1 = 2$ and $a_4 = 3$. Now, if $a_4 = 1$ or $a_3 = 4$, we will have a case 1 failure. We find that there are two such cases, so we have 24 situations in this case.

Now, we subtract back permutations we triple-counted. Since we already know there is no way to have two case 1 failures, we seek situations with a case 1 failure and two case 2 failures or three case 2 failures.

In the former case, consider our sample case: $a_1 = 2$, $a_4 = 3$, and $a_5 = 4$. If $a_2 = 1$ and $a_3 = 5$, we will also have a case 1 failure, while if $a_2 = 5$ and $a + 3 = 1$, we will not. Therefore, we have 6 failures in this case.

We can quickly observe that there are no situations where we have three case 2 failures.

So, our final answer is $44 - 20 - 48 + 12 + 24 - 6 = \boxed{6}$.

Because the problem involves so many cases, I wrote a Java program to confirm the answer. Email me at leedsjays@gmail.com if you'd like to see it. $\square$