2019 Practice Mathcounts Solutions

Austin Math Circle

January 20, 2019

Sprint Round

Problem 1. What is the sum of the first five odd numbers and the first four even numbers?

Solution. This is simply the sum of all the integers one through nine, which is $\boxed{45}$. □

Proposed by Jay Leeds.

Problem 2. Shipping a box costs a flat rate of \$5 plus \$2 for every pound after the first five pounds. How much does it cost to ship a 18-pound box?

Solution. An 18-pound box has 13 excess pounds, so our fee is $13 \cdot \$2 + \$5 = \boxed{\$31}$. □

Proposed by Jay Leeds.

Problem 3. In rectangle $ABCD$, $AB = 6$, $BC = 8$, and M is the midpoint of AB. What is the area of triangle CDM?

Solution. We see that this triangle has base six and height eight, so its area is $6 \cdot 8/2 = \boxed{24}$. □

Proposed by Jay Leeds.

Problem 4. Creed flips three coins. What is the probability that he flips heads at least once? Express your answer as a common fraction.

Solution. There is a $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$ chance that Creed flips no heads, so the probability that he flips at least once heads is $1 - \frac{1}{8} = \boxed{\frac{7}{8}}$. □

Proposed by Jay Leeds.

Problem 5. Compute the median of the following five numbers: $A = \frac{43}{9}$, $B = 4.5$, $C = \frac{23}{5}$, $D = 2^2$, and $E = 4.9$. Write A, B, C, D, or E as your answer.

Solution. We can easily sort $2^2 = 4$, 4.5, and 4.9. Dividing $\frac{23}{5}$ gives 4.6. After dividing out first two digits in $\frac{43}{9}$, we get 4.7, which clearly goes in between 4.6 and 4.9.

Therefore, the median is $\frac{23}{5}$, so the answer is $\boxed{C}$. □

Proposed by Josiah Kiok.

Problem 6. Josh wants to buy twenty widgets. Store A sells widgets for \$40 apiece, while Store B sells widgets for \$60 apiece. However, Store B is holding a sale: for every widget Josh buys from Store B, he gets an extra widget free. How much money would Josh save by buying the widgets at Store B instead of Store A?

Solution. At Store B, Josh can get two widgets for \$60, so he is paying \$30 per widget. Thus, he saves \$10 per widget, a total savings of $\boxed{\$200}$. □

Proposed by Jay Leeds.

Problem 7. A chemist dilutes five liters of 18% acid by adding four liters of water. What will be the concentration of acid in the resulting solution? Express your answer as a percentage.

Solution. With five liters of 18% acid, we have $5 \cdot \frac{18}{100} = \frac{9}{10}$ liters of acid in our solution. Dividing by our nine liters of solution, the concentration of acid is $\frac{1}{10} = \boxed{10\%}$. □

Proposed by Jay Leeds.

Problem 8. The cost of a dinner was supposed to be split evenly between Wayne, Xavier, Yanny, and Zed. However, Zed forgot to bring money, so the other three people shared the dinner bill equally among themselves. If each person had to pay \$13 more, how many dollars did the dinner cost?

Solution. Suppose the dinner cost D dollars. Then what was originally supposed to be $D/4$ dollars is now $D/3$ dollars, 13 more dollars than before. This gives us the equation

$$\frac{D}{3} - \frac{D}{4} = 13$$

The left hand side of the equation can be combined as $\frac{D}{3} - \frac{D}{4} = \frac{D}{12} = 13$, so the dinner cost $\boxed{156}$ dollars. □

Proposed by Jeffrey Huang.

Problem 9. Given that 673 is prime, what is the sum of the distinct prime factors of 20190?

Solution. We can easily see that $20190 = 10 \cdot 2019 = 2 \cdot 5 \cdot 2019$. Then, the divisibility test for three tells us that $2019 = 3 \cdot 673$, so our answer is $2 + 5 + 3 + 673 = \boxed{683}$. □

Proposed by Jay Leeds.

Problem 10. A miniature magic square is a two-by-two grid of squares such that the four cells contain the integers one through four, with each digit appearing once, and the two rows and two columns each contain digits summing to the same number. How many distinct miniature magic squares are there?

Solution. The answer is $\boxed{0}$. Because the sum of the four numbers is 10 and they will be split into two columns, each with the same column sum, each column and each row must have sum 5. Then, the number sharing a row with 1 must be $5 - 1 = 4$, but the number sharing a column with 1 must also be 4, which would require that 4 be in two places at once. Hence, it is impossible to construct a miniature magic square. □

Proposed by Jay Leeds.

Problem 11. Michael drives at 40 miles per hour on pavement and 20 miles per hour on dirt roads. He takes one hour to cross a 30-mile road made entirely of dirt and pavement. How many miles of this road are paved?

Solution. Let p be the paved length of the road. Since $d = rt$ (where d, r, and t are distance, rate, and time, respectively), we have that $t = \frac{d}{r}$, so writing an equation, we have $\frac{p}{40} + \frac{30-p}{20} = 1$. Simplifying, we have $60 - p = 40$, so $p = \boxed{20}$. □

Proposed by Jay Leeds.

Problem 12. Renee and Blake are putting red and blue beans into a pot, respectively. Currently, there are 13 red beans and 31 blue beans in the pot. If Renee wants at least two-fifths of the beans in the pot to be red, how many more red beans must she put into the pot?

Solution. Suppose that Renee puts in x more red beans, giving $13 + x$ red beans out of the total of $44 + x$ beans. By the problem's condition, Renee wants:

$$\begin{aligned} \frac{13+x}{44+x} &\geq \frac{2}{5} \\ 65 + 5x &\geq 88 + 2x \\ 3x &\geq 23 \\ x &\geq 8 - \frac{1}{3}. \end{aligned}$$

Because Renee can only put in an integer number of beans, she must put in at minimum $\lceil 8 - \frac{1}{3} \rceil = \boxed{8}$ red beans. □

Proposed by Nir Elber.

Problem 13. In a round-robin tournament, every team plays exactly one match against every other team. For each match, either one team wins and the other loses or both teams draw. A team that wins receives 2 points, a team that loses receives 0 points, and a team that draws receives 1 point. After all matches are played, the greatest number of points scored by a team is 10 points. What is the greatest possible number of teams that could have competed in the tournament?

Solution. If T teams play in the tournament, then a total of $T(T-1)$ points will be awarded, making for an average of $T-1$ points per team. A valid method for 11 teams would be to have all the teams draw, earning them all 10 points. If $T > 11$, then the average number of points per team would be greater than 10, which is not possible if 10 is the maximum possible score earned for any team. In conclusion, at most $\boxed{11}$ teams played in the tournament. □

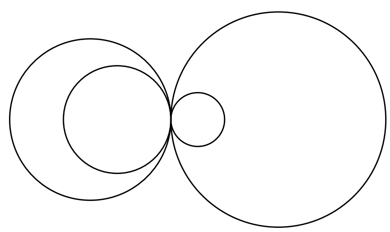
Proposed by Jeffrey Huang.

Problem 14. What is the remainder when $1+2+3+\cdots+2019$ is divided by 2024?

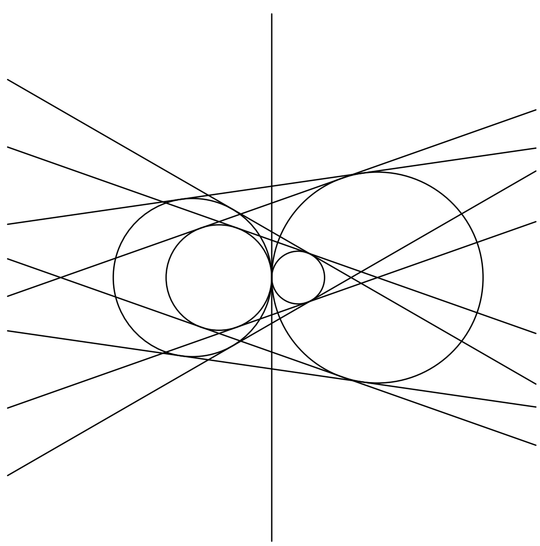
Solution. We get rid of all the pairs of terms that are divisible by 2024: $5+2019, 6+2018, 7+2017, \cdots, 1011+1013$. What remains is $1+2+3+4+1012 = \boxed{1022}$.
Alternate solution. Write $1+2+3+\cdots+2019 = \frac{2019\cdot 2020}{2} = 2019\cdot 1010 = (2024-5)\cdot 1010 \equiv -5\cdot 1010 = -5050 = -6072+1022 = -3\cdot 2024+1022 \equiv \boxed{1022}$. □

Proposed by Matthew Kroesche.

Problem 15. How many lines are tangent to at least two of the circles shown below?



Solution. There are $\boxed{9}$. They are shown in the figure below.



□

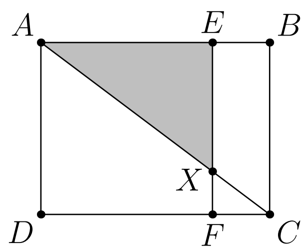
Proposed by Matthew Kroesche.

Problem 16. On January 1, Isaac poured 1000 liters of water into a giant beaker. On January N, where $N > 1$, Isaac extracted $\frac{1}{N}$ of the remaining amount of water in the giant beaker. At the end of January, how many liters of water remain in the giant beaker? Express your answer to the nearest whole number.

Solution. At the end of 31 days, the amount of water remaining in the beaker equals $1000 \times \frac{1}{2} \times \frac{2}{3} \times \ldots \times \frac{30}{31} = \frac{1000}{31}$, which is approximately $\boxed{32}$ liters. □

Proposed by Jeffrey Huang.

Problem 17. Rectangle $ABCD$ is defined such that $AB = 4$ and $BC = 3$. Points E and F lie on segments $\overline{AB}$ and $\overline{CD}$ respectively such that $AEFD$ forms a square. Let the intersection of segments $\overline{EF}$ and $\overline{AC}$ be X. Find the area of $\triangle AEX$. Express your answer as a common fraction.



Solution 1. For convenience, let $e = EX$ and $f = FX$. Because $\overline{AB} \parallel \overline{CD}$, we have $\angle EAX = \angle FCX$ and $\angle AEX = \angle CFX$, so $\triangle AEX \sim \triangle CFX$. This tells us that:

$$\frac{EX}{FX} = \frac{AE}{CF} \Rightarrow \frac{e}{f} = \frac{AE}{CF} = \frac{AE}{CD - DF} = \frac{3}{4-3} = 3,$$

where $AE = DF = 3$ because $AEFD$ is a square. Thus, $f = \frac{1}{3}e$. However, we also know that:

$$e + f = EX + XF = EF = 3,$$

once again because $AEFD$ is a square. Plugging in $f = \frac{1}{3}e$ into $e + f = 3$ gives us:

$$e + \frac{1}{3}e = 3 \Rightarrow e = \frac{9}{4}.$$

Finally, the area of right triangle $\triangle AEX$ is simply $\frac{AE \cdot EX}{2} = \frac{AE \cdot e}{2} = \frac{3 \cdot 9/4}{2} = \boxed{\frac{27}{8}}$. □

Solution 2. Observe that segments $\overline{EX}$ and $\overline{BC}$ are perpendicular to $\overline{AB}$ ($AEFD$ is a square, and $ABCD$ is a rectangle), so we know $\angle AEX = \angle ABC = 90°$. But obviously, $\angle EAX = \angle BAC$, so $\triangle AEX \sim \triangle ABC$ by Angle-Angle similarity. Now, because $\triangle AEX \sim \triangle ABC$, we know that:

$$\frac{[AEX]}{[ABC]} = \left(\frac{AE}{AB}\right)^2 = \left(\frac{3}{4}\right)^2 = \frac{9}{16},$$

where $AE = AD = 3$ because $AEFD$ is a square. Additionally, we know that $\triangle ABC$ is a right triangle, so its area is just $[ABC] = \frac{AB \cdot BC}{2} = \frac{3 \cdot 4}{2} = 6$. It follows that:

$$[AEX] = \frac{9}{16} \cdot [ABC] = \frac{9}{16} \cdot 6 = \boxed{\frac{27}{8}},$$

which matches the answer found in the first solution. □

Proposed by Nir Elber.

Problem 18. For all real numbers $a, b \geq 0$, define the operation $a \odot b$ as

$$a \odot b = a + b + 2\sqrt{ab}$$

Evaluate

$$((\cdots(((1^2 \odot 2^2) \odot 3^2) \odot 4^2) \cdots) \odot 19^2)$$

Solution. Observe that $a \odot b = \left(\sqrt{a} + \sqrt{b}\right)^2$, so $(a \odot b) \odot c = \left(\sqrt{(\sqrt{a} + \sqrt{b})^2} + \sqrt{c}\right)^2 = (\sqrt{a} + \sqrt{b} + \sqrt{c})^2$. By repeatedly applying this formula, the expression we are asked to calculate reduces to $\left(\sqrt{1^2} + \sqrt{2^2} + \cdots + \sqrt{19^2}\right)^2 = (1 + 2 + \cdots + 19)^2 = \left(\frac{19 \cdot 20}{2}\right)^2 = 190^2 = \boxed{36100}$. □

Proposed by Matthew Kroesche.

Problem 19. A semi-prime is a number that can be written as pq for not necessarily distinct primes p and q. Find the sum of all the factors of 1400 that are semi-prime.

Solution. Observe that $1400 = 2^3 \cdot 5^2 \cdot 7$, so all factors will be of the form $2^a \cdot 5^b \cdot 7^c$ for nonnegative integers a, b, and c. Because we only care about semi-prime factors, the factors we care about turn out to be either squares of primes or products of two distinct primes. However, we must be a bit careful when summing because 7^2 is not actually a factor of 1400. If we naively include 7^2, we are asked to find:

$$S = 2^2 + 5^2 + 7^2 + 2 \cdot 5 + 5 \cdot 7 + 7 \cdot 2. \tag{$*$}$$

We are trying to find $S - 7^2$. Now, we could run the arithmetic on $(*)$, or we can write it as one of the following:

$$S = \frac{1}{2}\left((2+5)^2 + (5+7)^2 + (7+2)^2\right) = (2+5+7)^2 - 2 \cdot 5 - 5 \cdot 7 - 7 \cdot 2 = \frac{1}{2}\left((2+5+7)^2 + 2^2 + 5^2 + 7^2\right),$$

which may be marginally easier. For example, $S - 7^2 = \frac{1}{2}\left(14^2 + 2^2 + 5^2 - 7^2\right) = \frac{1}{2}(196 - 20) = \frac{1}{2}(176) = \boxed{88}$.
Comment: The straight arithmetic is not so bad: We want $2^2 + 5^2 + 2 \cdot 5 + 5 \cdot 7 + 7 \cdot 2 = 4 + 25 + 10 + 35 + 14 = 88$. □

Proposed by Nir Elber.

Problem 20. Find the probability that a randomly chosen three-digit number $\overline{ABC}$ with $A \neq 0$ satisfies the equation $A = B + C$. Express your answer as a common fraction.

Solution. For all three-digit numbers $\overline{ABC}$, A, B, and C must be nonnegative integers less than 9, and A must be greater than 1. This gives $9 \cdot 10 \cdot 10 = 900$ total three-digit numbers. However, for three-digit numbers satisfying $A = B + C$, notice that for any digits A and $B \leq A$, there exists a unique digit C satisfying $A = B + C$ and thus a unique $\overline{ABC}$ satisfying $A = B + C$. Thus, we can count the number of "good" $\overline{ABC}$ by counting the number of $B \leq A$ for digits A. At this point, we can run generalized casework on the values of A: If $A = k$, then there are $k + 1$ possible values of B because $B \in \{0, 1, 2, \ldots, k\}$. So, the number of three-digit numbers is:

$$\underbrace{2}_{A=1} + \underbrace{3}_{A=2} + \ldots + \underbrace{10}_{A=9} = (1 + 2 + \ldots + 9 + 10) - 1 = \frac{10 \cdot 11}{2} - 1 = 54.$$

The desired probability is thus $\frac{54}{900} = \boxed{\frac{3}{50}}$. □

Proposed by Nir Elber.

Problem 21. Compute the sum of the digits of

$$9 + 99 + 999 + 9999 + \cdots + 99999999999999999999$$

where the last term has 20 9's.

Solution. We write this as

$$(10^1 - 1) + (10^2 - 1) + \cdots + (10^{20} - 1) = (10^1 + 10^2 + \cdots + 10^{20}) - 20$$

The first term is simply twenty 1's followed by a 0, so we keep the first 18 1's and write the last three digits as $110 - 020 = 090$. Thus, all in all, the digit sum is $18 \cdot 1 + 9 = \boxed{27}$. (Written explicitly, this number is 11111111111111111090.) $\square$

Proposed by Matthew Kroesche.

Problem 22. Alex is reading a 219-page book. On the first day, he reads 3 pages, and on every following day, he reads two pages more than the day before. For example, he reads five pages on the second day, seven on the third day, etc. (He may read fewer than the prescribed number of pages on the day he finishes the book.) How many pages will Alex read on the final day?

Solution. This is a very slight variation on the classic finite differences problems.

Notice that on any day n, Alex is reading $1+2n$ pages, so at the end of day n, Alex would have read the following.

$$\begin{aligned}
\underbrace{3+5+7+\ldots+1+2n}_{n\text{ terms}} &= \underbrace{1+1+1+\ldots+1}_{n\text{ terms}} + \underbrace{2+4+6+\ldots+2n}_{n\text{ terms}} \\
&= \qquad n \qquad + 2(1+2+3\ldots+n) \\
&= n + 2\left(\frac{n(n+1)}{2}\right) \\
&= n^2 + 2n.
\end{aligned}$$

Experienced problem-solvers may immediately recognize this as an application of summing the first few n odd numbers, skipping over the 1. In particular, we are able to rewrite the number of pages Alex has read at the end of day n as

$$(n+1)^2 - 1.$$

Observe that Alex on the last day (call it day N) would have read $(N+1)^2 - 1$ pages, but it turns out that $(N+1)^2 - 1 \geq 219$, so Alex instead reads $(N+1)^2 - 220 \geq 0$ pages on that day. The smallest value of N that makes this value positive is $N = 14$, so indeed, $(14+1)^2 - 220 = 5$ is the number of pages Alex doesn't read on that day, so $2 \cdot 14 + 1 - 5 = \boxed{24}$ is our answer. (After $N = 14$, Alex should have read $15^2 - 1 = 224 > 219$ pages and so has already finished the book, so $N = 14$ is indeed the day that Alex finishes the book.) $\square$

Proposed by Nir Elber.

Problem 23. Raymond writes a three-digit number and its reverse, neither of which have leading zeroes, on a blackboard. (For example, 148 and 841 are reverses of each other.) Edward calculates the sum of the two numbers Raymond has written, and finds that it is equal to 1534. What is the smallest possible value of either of the numbers Raymond could have written?

Solution. If one of Raymond's numbers was $\underline{A}\,\underline{B}\,\underline{C} = 100A + 10B + C$, then the other was $\underline{C}\,\underline{B}\,\underline{A} = 100C + 10B + A$, and so their sum is $101A + 20B + 101C = 101(A+C) + 20B = 1534$. Thus, the units digit of this expression is the same as the units digit of $A + C$, which is 4. Since $A + C \leq 9 + 9 = 18$, we see that $A + C$ is either 4 or 14, and if it's equal to 4, the sum is at most $404 + 20 \cdot 9 = 584$ which is too small. Thus $A + C = 14$, and $101 \cdot 14 + 20B = 1534 \implies 20B = 1534 - 1414 = 120 \implies B = 6$. Then the smallest positive integer with $A + C = 14$ and $B = 6$ occurs when $C = 9$ and $A = 5$, so the answer is $\boxed{569}$. Sure enough, $569 + 965 = 1534$. $\square$

Proposed by Matthew Kroesche.

Problem 24. In a dice game, Albert rolls an eight-sided die numbered with the integers from one to eight while Josiah rolls a ten-sided die numbered with the integers from one to ten. Whichever of the two rolls a higher number wins the game. If both of them roll the same number, they roll again until one of them wins. What is the probability that Albert will win this game?

Solution. Albert wins whenever he rolls a number and Josiah rolls a lower one. If Albert rolls a one, there is no way he could win on that roll. If he rolls a two, there is one way he could win (if Josiah rolls a one). If Albert rolls a three, there are two ways he could win: Josiah must roll one or two. This continues until Albert has seven ways to win if he rolls an eight. Hence, 28 rolls lead to a victory for Albert.

Now, we consider how many rolls will end the game. There are 80 possible rolls in total, but 8 of them lead to ties, so there are 72 game-ending rolls. Hence, our answer is $\frac{28}{72} = \boxed{\frac{7}{18}}$. As a quick check, it seems that Albert's probability should be a little under one-half, so this answer seems plausible.

Alternative Solution. We split into three cases. First, Albert and Josiah could roll two different numbers that are both at most eight. There are seven numbers Josiah could roll to give this result no matter what Albert rolls, so this happens $\frac{7}{10}$ of the time. In this case, Albert's number will be higher half the time, so Albert wins on $\frac{7}{20}$ of rolls and loses on another $\frac{7}{20}$. Second, Albert and Josiah could tie, which clearly happens $\frac{1}{10}$ of the time. Finally, Josiah could roll a nine or a ten, which happens $\frac{1}{5}$ of the time.

Then, let p be the probability Albert wins. We now know that $p = \frac{7}{20} + \frac{p}{10}$, because on each roll, Albert wins $\frac{7}{20}$ of the time and has a $\frac{1}{10}$ chance of forcing a reroll, which keeps his probability equal to p. Solving this equation gives $p = \boxed{\frac{7}{18}}$. □

Proposed by Jay Leeds.

Problem 25. On Mars, there are four types of coins: yellow coins, which are worth six cents, and red, green, and blue coins, which are each worth one cent. How many different sets of Martian coins are worth twenty cents in total? (Two sets are different only if they contain different numbers of some colored coin.)

Solution. We do casework on the number of yellow coins. If there are y yellow coins, then we need to select $20 - y$ red, green, or blue coins. By the Hockey Stick Identity, there are $\binom{22-6y}{2}$ ways to do so.

Hence, since y can be any number zero through three, our answer is:

$$\binom{22}{2} + \binom{16}{2} + \binom{10}{2} + \binom{4}{2} = 231 + 120 + 45 + 6 = \boxed{402.}$$

□

Proposed by Jay Leeds.

Problem 26. Let $x = 0.201920192019\ldots$. We construct a real number y so that the n^{th} decimal place of y is the $n!^{\text{th}}$ decimal place of x. Compute y. Express your answer as a common fraction.

Solution. Construct the sequence "d_n" such that $d_0 = 2, d_1 = 0, d_2 = 1, d_3 = 9$ and $d_n = d_{n \bmod 4}$ so that we can write $x = 0.d_0d_1d_2\ldots$. We can see that the n^{th} decimal place of x is d_{n-1}, so the $n!^{\text{th}}$ decimal place of x is $d_{n!-1}$. It follows that

$$y = 0.d_{1!-1}d_{2!-1}d_{3!-1}d_{4!-1}d_{5!-1}d_{6!-1}\ldots.$$

Notice that for $k \geq 4$, we know that $k! - 1 \equiv 0 - 1 \equiv 3 \pmod 4$, so $d_{k!-1} = d_3 = 4$. Thus, we can fully write out y:

$$y = 0.d_0d_1d_5d_3d_3d_3\ldots = 0.200\overline{9}.$$

Thus, $1000y = 200 + \frac{9}{9} = 201$, so $y = \boxed{\frac{201}{1000}}$. □

Proposed by Nir Elber.

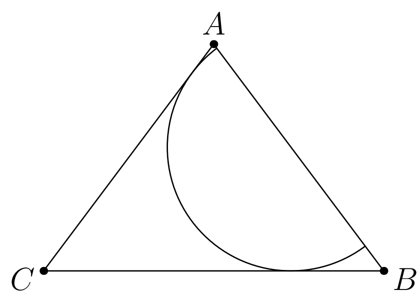
Problem 27. Call an integer *squareful* if it can be written as the sum of some (not necessarily distinct) perfect squares greater than 1. Compute the largest positive integer which is not squareful.

Solution. By the Frobenius Coin Theorem (also known as the Chicken McNugget Theorem), the greatest number that cannot be expressed as the sum of 4 and 9 is $4 \cdot 9 - 4 - 9 = 23$. Therefore, the answer is at most 23. The only square other than 4 or 9 that's less than 23 is 16, but since $16 = 4 \cdot 4$, adding sixteen is equivalent to adding four four times, so it won't give us any new options to construct 23. Therefore, there is no way to express 23 as the sum of squares greater than one, so $\boxed{23}$ is the largest number that is not squareful.

□

Proposed by Nir Elber.

Problem 28. A semicircle is inscribed in isosceles $\triangle ABC$ such that the center lies on $\overline{AB}$ and the semicircle is tangent to both other sides of the triangle. Given that the $AB = AC = 5$ and $BC = 6$, find the radius of the semicircle. Express your answer as a common fraction.



Solution. Let the center of the semicircle be I, and let its radius be r. Call the points of tangency to $\overline{BC}$ and $\overline{AC}$ be D and E respectively. Observe that for tangency reasons, $\overline{ID} \perp \overline{BC}$ and $\overline{IE} \perp \overline{AC}$. It follows that:

$$[\triangle ABC] = [\triangle AIC] + [\triangle BIC] = \frac{AC \cdot IE}{2} + \frac{BC \cdot ID}{2} = \frac{5r}{2} + \frac{6r}{2} = \frac{11r}{2}, \qquad (*)$$

where $[\triangle XYZ]$ denotes the area of $\triangle XYZ$. Thus, it suffices to find the area of $\triangle ABC$, doable without Heron's because $\triangle ABC$ is isosceles. Indeed, let the midpoint of $\overline{BC}$ be M, so $\overline{AM} \perp \overline{BC}$ because $\triangle ABC$ is isosceles. Now,

$$AM = \sqrt{AB^2 - BM^2} = \sqrt{5^2 - 3^2} = 4$$

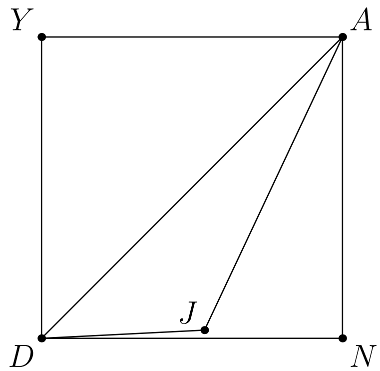
because $\triangle AMB$ is a right triangle. It follows that $[\triangle ABC] = \frac{BC \cdot AM}{2} = \frac{6 \cdot 4}{2} = 12$. Comparing this with $(*)$, we know $\frac{11r}{2} = 12$ so $r = \boxed{\frac{24}{11}}$.

Comment: This problem is really an exercise to see if students understand the $A = rs$ formula; the calculation of $[\triangle ABC]$ should be routine. □

Proposed by Nir Elber.

Problem 29. Point J lies in the interior of square $ANDY$ such that $AJ = 8$ and $JD = 4$. Compute the least possible integer value for the area of square $ANDY$.

Solution. Triangle ADJ has its third side AD as the diagonal of $ANDY$. We want to minimize the length of AD since the area is simply $\frac{AD^2}{2}$, but we cannot make it too low because point J must remain inside the square. Specifically, the smaller we make AD, the bigger angle $\angle ADJ$ gets, until eventually it hits 45°, at which time J moves across side DN. At the instant when this happens, the square has side length s. By the Pythagorean Theorem, we have $AN^2 + NJ^2 = AJ^2 \implies s^2 + (s-4)^2 = 8^2$. This quadratic simplifies to $2s^2 - 8s - 48 = 0 \implies s^2 - 4s - 24 = 0$. Then by the quadratic formula, $s = \frac{4\pm\sqrt{16+96}}{2} = 2+2\sqrt{7}$. (The $\pm$ must be a $+$ so that s will be positive.) Then the area of square $ANDY$ is $s^2 = 4+28+8\sqrt{7} = 32+\sqrt{448}$. Since $21^2 < 448 < 22^2$, the least integer area bigger than that is $32+22 = \boxed{54}$.



$\square$

Proposed by Matthew Kroesche.

Problem 30. In the following addition problem, each letter represents a distinct nonzero digit. Compute the least possible value of the four-digit number $MATH$.

$$\begin{array}{rcccccc} & A & U & S & T & I & N \\ + & & & M & A & T & H \\ \hline & 2 & 0 & 1 & 9 & 0 & 0 \end{array}$$

$$\begin{array}{rcccccc} & 1 & 9 & 8 & 7 & 2 & 6 \\ + & & & 3 & 1 & 7 & 4 \\ \hline & 2 & 0 & 1 & 9 & 0 & 0 \end{array}$$

Solution. We start at the right and work our way left. First we have that $N+H = 10$, because 0 is too small and 20 is too large. Thus a 1 is carried into the tens column, so $1+I+T$ ends in 0. Then $I+T = 9$ because 19 is too big. So a 1 is carried into the hundreds column, and then $1+T+A$ ends in 9. So $T+A = 8$ because 18 would require $T = A = 9$. Then nothing is carried into the thousands column, and so $S+M$ is 11, because 1 would require one of S, M to be zero, and 21 is too large. Then $U+1$ ends in 0, so $U = 9$, and $A+1$ ends in 2, so $A = 1$. Since $T+A = 8$, $T = 7$, and since $I+T = 9$, $I = 2$. All we have left, then, is $S+M = 11$ and $N+H = 10$, and the digits we haven't used are $3,4,5,6,8$. Thus we need $N, H = 4,6$ in some order, and $S, M = 3,8$ in some order. To minimize the value of $MATH$, we choose $H = 4$ and $M = 3$, so our answer is $\boxed{3174}$.

Comment: This is still the answer if we ignore the restraint that all digits must be nonzero, but it just makes it slightly harder to show that it is the answer. $\square$

Proposed by Matthew Kroesche.

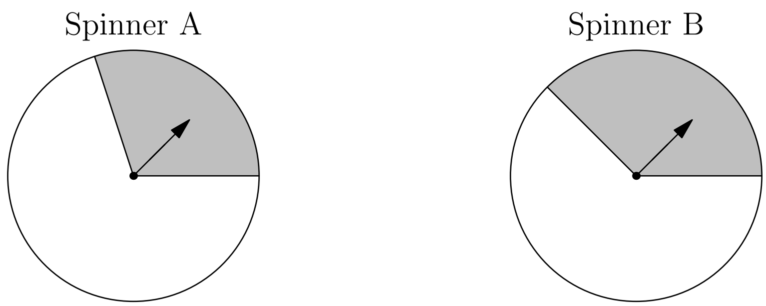
Target Round

Problem 1. When exposed to the sun, a sample of a certain liquid becomes 20% smaller every hour. After how many whole hours will the volume of the sample first drop below 10% of its original size?

Solution. After one hour, the volume of the sample is 0.8 of its original value. After two hours, the sample reaches 0.64 of its original size. Continuing along similar lines, after ten hours, the sample's volume is just 10.7% of its original value, so it drops below 10% after $\boxed{11}$ hours. □

Proposed by Jeffrey Huang.

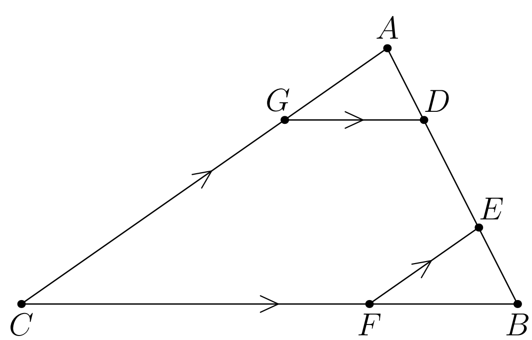
Problem 2. Spinners A and B are constructed, as shown, so that the radii drawn in Spinner A are sides of a regular pentagon, and the radii drawn in Spinner B are sides of a regular octagon. If both arrows are spun, what is the probability that exactly one of them lands in its spinner's shaded region? Express your answer as a common fraction.



Solution. The angle of a regular pentagon is 108 degrees, so the probability of landing in the shaded region is $\frac{108}{360} = \frac{3}{10}$. The angle of a regular octagon is 135 degrees, so the probability of landing in the shaded region is $\frac{135}{360} = \frac{3}{8}$. The probability of landing in exactly one shaded region is $\frac{3}{10} \times \frac{5}{8} + \frac{7}{10} \times \frac{3}{8} = \frac{15}{80} + \frac{21}{80} = \frac{36}{80} = \boxed{\frac{9}{20}}$. □

Proposed by Jeffrey Huang.

Problem 3. In triangle ABC, points D and E are on side AB, point F is on side BC, and point G is on side AC. Additionally, EF is parallel to AC and DG is parallel to BC. If angle ADG has measure 63°, while angle BEF has measure 82° , find the measure of angle ACB.



Solution. Because DG is parallel to BC, we have that angle EBF is equal to 63°. Moreover, we have that $\angle BFE = 180° - \angle BEF - \angle EBF = 180° - 145° = 35°$. Then, because EF is parallel to AC, we have that $\angle ACB = \angle BFE = \boxed{35°}$. □

Proposed by Jay Leeds.

Problem 4. The base −4 representation of $23 = 2 \cdot (-4)^2 + 3 \cdot -4 + 3$ is 233_{-4}. Find the base −3 representation of 23.

Solution. We do casework on the number of digits in 23_{-3}. The maximum three-digit base -3 number is $2\cdot(-3^2)+0\cdot -3+2=20$, so we'll need more digits than that. $(-3)^3$ is negative, so the fourth digit doesn't help, but the fifth digit will add $(-3)^4=81$, which might help us. We note that the last four digits will never change the magnitude of the total by more than 81, so the first digit needs to be a 1. Similarly, we conclude that the next digit needs to be a 2, to bring our running total down to $81+2\cdot -27=27$. Now, we need to construct a -4 with the final three digits. The next digit, a 9, is positive, so we don't add any of those. The next digit, -3, is negative, so we add two of those to bring our running total to 21, since the final value, 1, is positive. We need to add 2 ones to get to a total of 23, so our answer is $\boxed{12022_{-3}}$. Checking shows that this is equal to $81-54-6+2=23$, so this indeed works. □

Proposed by Jay Leeds.

Problem 5. Carol the Curator is counting coconuts in her cubic closet. Each coconut is a unit cube, and her closet's length, in units, is an integer l. She tries putting the coconuts in three long rows, each with dimensions $1\times 1\times l$, but she finds that she has 29 coconuts left over. She then fills the entire floor of the closet, which has dimensions $1\times l\times l$, with coconuts, only to find that she still has one coconut remaining. How many coconuts is Carol counting?

Solution. Let the number of coconuts Carol is counting be c and the side length of her cubic closet be s. The problem gives the following system:

$$\begin{cases} c-3s=29 \\ c=s^2+1 \end{cases}.$$

Plugging in for c in the second equation tells us:

$$\begin{aligned} 3s+29 &= s^2+1 \\ 0 &= s^2-3s-28 \\ 0 &= (s-7)(s+4), \end{aligned}$$

and because Carol's room has a positive side length, $s=7$ units. Thus, $c=s^2+1=7^2+1=\boxed{50}$. □

Proposed by Nir Elber.

Problem 6. A "substring" of a positive integer n is any group of consecutive digits of n. For example, 1, 23, and 123 are substrings of 123, but 13 is not. Find the largest positive integer with distinct digits such that all of its substrings are prime numbers.

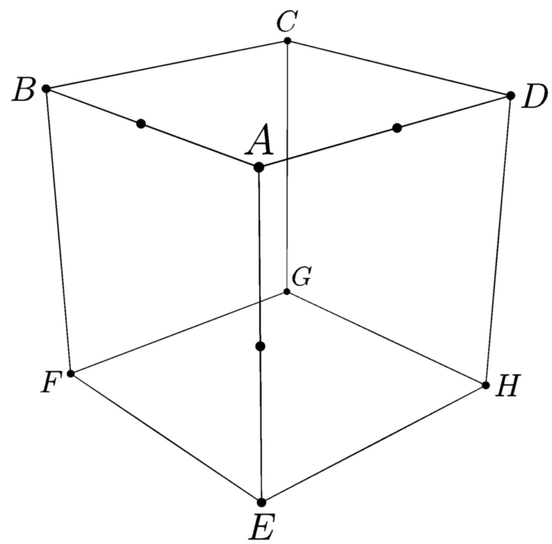
Solution. For convenience, let the largest number be n. Obviously, all substrings of length 1 in n are individual digits, so each digit of n must be prime: 2, 3, 5, or 7. Because n has distinct digits, we know n has four or fewer digits. Consider the primes 2 and 5: Any substring of length greater than 1 ending in 2 or 5 will be divisible by 2 or 5 respectively and so not be prime. Thus, if one of 2 or 5 occur in n, then it be the first digit of n so that there is no substring of length greater than 1 ending in 2 or 5. Observe that both 2 and 5 cannot occur in n because both cannot the first digit; thus, n is not four digits long.

If n has three digits, n must contain the digits $\{3,5,7\}$ or $\{2,3,7\}$ because 2 and 5 cannot occur simultaneously. However, the digit sum of first triplet is $15=3\cdot 5$, and the digit sum of the second triplet is $12=3\cdot 4$, so no matter how those groups of digits are arranged in n, n itself (which is a substring) will be divisible by 3 and so not be prime.

Finally, we try n that have two digits. Testing two-digit numbers composed of distinct primes, 75 is the largest but obviously fails; however, 73 works. We conclude that $\boxed{73}$ is our answer. □

Proposed by Nir Elber.

Problem 7. In a unit cube ABCDEFGH, a single slice is made across the midpoints of AB, AD, and AE to form two solids. What is the volume of the larger solid, in cubic units? Express your answer as a common fraction.



Solution. We instead compute the volume of the smaller solid, which we compute using the pyramid volume formula. The volume of the smaller solid is the area of the base, which is $\frac{1}{8}$, times the height, which is $\frac{1}{2}$, then divided by 3, which equals $\frac{1}{48}$. Hence, the volume of the larger solid is $\boxed{\frac{47}{48}}$ cubic units. □

Proposed by Jeffrey Huang.

Problem 8. At the Slightly Pointless Ultimate Regional Smackdown (SPURS), there are 100 contestants, numbered 1 through 100. The contestants all take a 30-problem test, with problems numbered 1 through 30. For each m and n, the probability that contestant m solves problem n is $\frac{m}{mn+60}$. (All these probabilities are assumed to be independent.) What is the probability that every contestant solves at least one problem on the test? Express your answer as a common fraction.

Solution. The probability that contestant m does not solve problem n is $\frac{m(n-1)+60}{mn+60}$. Thus, the probability that contestant m does not solve a single problem on the entire test is

$$\frac{60}{m+60}\cdot\frac{m+60}{2m+60}\cdots\frac{29m+60}{30m+60}=\frac{60}{30m+60}=\frac{2}{m+2}$$

Then the probability that contestant m solves at least one problem is $1-\frac{2}{m+2}=\frac{m}{m+2}$. Finally, we find the probability that every contestant solves at least one problem. This is

$$\frac{1}{3}\cdot\frac{2}{4}\cdot\frac{3}{5}\cdots\frac{100}{102}=\frac{1\cdot 2}{101\cdot 102}=\frac{1}{101\cdot 51}=\boxed{\frac{1}{5151}}$$

□

Proposed by Matthew Kroesche.

Team Round

Problem 1. The average temperature of a ring in degrees Fahrenheit is twice the temperature in degrees Celsius. The average temperature of the ring is how many degrees Celsius? (The temperature F in degrees Fahreinheit satisfies the equation $F = \frac{9}{5}C + 32$, where C is the equivalent temperature in degrees Celsius.)

Solution. Using the formula $F = \frac{9}{5}C + 32$, we find that $\frac{9}{5}C + 32 = 2C$, giving $\frac{1}{5}C = 32$, giving $C = \boxed{160}$ degrees. □

Proposed by Jeffrey Huang.

Problem 2. Triangle ABC is inscribed in a circle with AB as a diameter. Given that minor arc $\widehat{BC}$ has measure $130°$, find $\angle ABC$.

Solution. Since AB cuts off an arc of $180°$ (because it is a diameter), we conclude by the inscribed angle theorem that $\angle C = 90°$. Similarly, since $\widehat{BC}$ has measure $130°$, we conclude that $\angle A$ has measure $\frac{130°}{2} = 65°$. Hence, $\angle B = 180° - 90° - 65° = \boxed{25°}$. □

Proposed by Jay Leeds.

Problem 3. On his first three tests in a class, Robert earns grades of $70, 80$, and 90 points. After Robert takes a fourth test, the median of his four grades is m. Given that m is an integer, what is the sum of the possible values of m?

Solution. In a set of four data points, the median is the average of the second and third values. Let g be Robert's fourth test score. If $g < 70$, the median will be 75, and likewise, if $g > 90$, the median will be 85. If $70 < g < 90$, 80 and g will be the middle values, so the median is $\frac{80+g}{2}$. This can take on any of the integer values between 75 and 85. So, our answer is the sum of the integers from 75 to 85, and it is easy to see that this is a set of eleven numbers averaging 80, so our answer is $\boxed{880}$. □

Proposed by Jay Leeds.

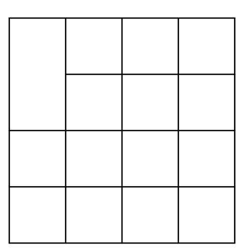
Problem 4. When Samantha's teacher asked her to compute the least common multiple of n and $20 - n$ for a given integer n, she computed the least common positive *factor* instead. Given that Samantha's answer was 50 less than what is should have been, compute the largest possible n.

Solution. Obviously, the least common factor between any two integers is 1, so the least common multiple of n and $20 - n$ is actually $1 + 50 = 51$. We know that, by definition of the lcm, n and $20 - n$ must be divisors of 51, and because $51 = 3 \cdot 17$, the only possible divisors of 51 are 1, 3, 17, and 51.

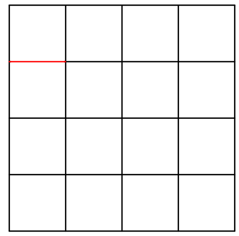
More or less, we are now looking for divisors of 51 that sum to $n + 20 - n = 20$. Just looking at the divisors (or by quick trial-and-error), we know that only $(n, 20 - n) = (3, 17)$ or $(17, 3)$ will work, which give $n = 3$ or $n = 17$. Thus, the largest possible n is $\boxed{17}$. □

Proposed by Nir Elber.

Problem 5. How many rectangles can be formed in the following grid, given that all rectangles have vertices at the intersection of 2 grid lines and sides parallel to grid lines as well?



Solution. We begin by completing the grid and will finish with complementary counting.



We will now use complementary counting. To count the number of rectangles in the grid, we remark that any rectangle is uniquely defined by choosing 1 pair of horizontal grid lines and 1 pair of vertical grid lines. Thus, there are

$$\frac{5\cdot 4}{2}\cdot\frac{5\cdot 4}{2}=100$$

total rectangles. Now we count the number of rectangles using the red line in order to subtract them out. Observe that using the red line forces the rectangle to use that horizontal grid line but also the leftmost vertical grid line; else the rectangle will miss that grid line. Thus, we really only have choice about the second grid line in each pair; that is, there are

$$4\cdot 4=16.$$

rectangles that use the red line. We conclude that our answer is $100-16=\boxed{84}$ rectangles. □

Proposed by Nir Elber.

Problem 6. Every day, the probability of no rain in Rain Land is q times the probability of rain. If the probability of rain is y, and $q+y=9$, on any day, what is the probability that Rain Land will be raining? Express your answer as a decimal to the nearest thousandth.

Solution. We only need to solve for y. Notice that if y is the probability of rain, then $1-y$ is the probability of no rain. Then $\frac{1-y}{y}=q$, so $q+y=\frac{1-y}{y}+y=\frac{y^2-y+1}{y}=9$. Multiplying both sides by 9, we get $y^2-y+1=9y$, or $y^2-10y+1=0$. Solving for y, we find that $y=5\pm 2\sqrt{6}$. Since y must be in the interval $[0,1]$, we must have $y=5-2\sqrt{6}$. Hence, y is approximately $\boxed{0.101}$. □

Proposed by Jeffrey Huang.

Problem 7. Jay can buy two bagels and one muffin with exactly twenty silver coins. Kevin can buy one bagel and two muffins with exactly three gold coins. Together, they can buy five bagels and seven muffins with exactly eleven gold coins and four silver coins. If a gold coin is worth n times as much as a silver coin, compute n.

Solution. Let b and m be the cost of a bagel and a muffin, respectively, in silver coins. Then

$$2b+m=20$$

$$b+2m=3n$$

$$5b+7m=(2b+m)+3(b+2m)=20+9n=11n+4$$

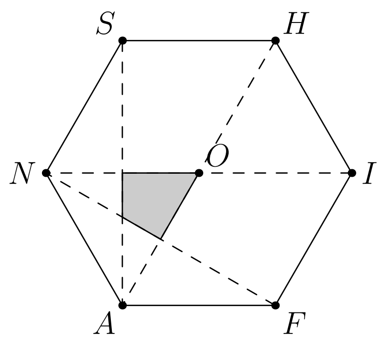
$$2n=16\implies n=\boxed{8}$$

□

Proposed by Matthew Kroesche.

Problem 8. Regular hexagon $SHIFAN$ has area 1. Compute the area of the intersection of triangles ASH and FIN. Express your answer as a common fraction.

Solution. Let O be the center of hexagon $SHIFAN$. The area of the intersection is contained within equilateral triangle ANO, which has area $\frac{1}{6}$. Triangles SAN and FAN are each isosceles, with two 30° angles and one 120° angle. Then since $\angle ANO = \angle NAO = 60°$, NF bisects $\angle ANO$ and AS bisects $\angle NAO$. Thus, since ANO is equilateral, AS is the perpendicular bisector of NO and NF is the perpendicular bisector of AO. So the area is the region of ANO that lies on the same side of the perpendicular bisectors of NO and AO as O, which is equivalently the set of points in ANO that are closer to O than any other vertex. This is $\frac{1}{3}$ of the area of ANO, which is thus $\frac{1}{3} \cdot \frac{1}{6} = \boxed{\frac{1}{18}}$.



□

Proposed by Matthew Kroesche.

Problem 9. In the sequence a_1, a_2, a_3, ..., the term a_n is equal to $0.N$, where N is written as exactly the positive integer N with the digits in that order (no leading zeros). What is the greatest possible positive value of $a_{n+2019} - a_n$ less than 0.2019 over all positive integers n? Express your answer as a decimal to the nearest ten-thousandth.

Solution. If n and $n + 2019$ have the same number of digits, then the result is going to 0.2019 multiplied by a nonnegative integer power of 0.1. The largest of those values less than 0.2019 is 0.02019.

Across all one-digit integers n, adding every time we increment n by 1, we increment a_{n+2019} by 0.0001 and a_n by 0.1, making our desired difference smaller. Hence, the largest possible difference is $0.2020 - 0.1 = 0.1020$. For n a two-digit integer, our optimal difference is $0.2029 - 0.10 = 0.1029$. For three-digit integers, our optimal difference is $0.2119 - 0.100 = 0.1119$.

Otherwise, n contains at least 4 digits, and the only scenario in which n and $n+2019$ have a different number of digits is if $n+2019$ has exactly one more digit than n. In each of those cases, incrementing $n+2019$ by 1 increments a_{n+2019} by a power of 10 that is less than the power of 10 we increment a_n by when we increment n by 1. Hence, we would like to minimize n in these cases to maximize the difference $a_{n+2019} - a_n$. This happens when $n+2019$ is a power of 10. We would then have $a_{n+2019} = 0.1$, but if $a_n < a_{n+2019}$, then a_n must have its tenths digits be 0, but N cannot have leading zeros. It follows that if $n + 2019$ has exactly one more digit than n for n large enough, then $a_{n+2019} - a_n \leq 0$.

Examining everything we have worked through, it follows that the largest possible value of $a_{n+2019} - a_n$ is $\boxed{0.1119}$. □

Proposed by Jeffrey Huang.

Problem 10. A class of 7 distinguishable students taking a math test is ordered randomly in a line. Matthew knows the answers to half of the questions, and Jeffrey knows the answers to the other half. Three other students cheat by copying answers from students to their left or right, and the last two students answer nothing. If cheaters only cheat off of Matthew, Jeffrey, and other cheaters, compute the probability all cheaters score perfectly. Express your answer as a common fraction.

Solution. Label Matthew by an $\mathbb{M}$, Jeffrey by a $\mathbb{J}$, the cheaters by C_1, C_2, and C_3, and the students who answer none by N_1 and N_2. In this way, we can describe all seatings; for example, $N_1\mathbb{J}C_2C_1\mathbb{M}C_3N_2$ is such an arrangement. Because students are distinguishable, there are 7! such arrangements. It now suffices to count the arrangements in which the cheaters get a perfect score.

We claim the cheaters get a perfect score iff there is a continuous block of all C_i bordered by $\mathbb{M}$ and $\mathbb{J}$.

First, this is sufficient. If a block of C_i is found between $\mathbb{M}$ and $\mathbb{J}$ (C_1C_2 in $N_1\mathbb{J}C_2C_1\mathbb{M}C_3N_2$), then those C_i will get a perfect score: From one side, Matthew's answers will propagate across the block, and the reverse happens on the other side with Jeffrey's answers.

Second, this is necessary. If any C_i is found outside such a block (C_3 in $N_1\mathbb{J}C_2C_1\mathbb{M}C_3N_2$), then that C_i will be unable to receive all of the answers because answers can't pass through $\mathbb{M}$ and $\mathbb{J}$. (Answers from $\mathbb{J}$ cannot pass through $\mathbb{M}$ to C_3 in $N_1\mathbb{J}C_2C_1\mathbb{M}C_3N_2$.) Additionally, if a string of C_i between $\mathbb{M}$ and $\mathbb{J}$ is discontinuous/interrupted by an N_j ($\mathbb{M}C_3N_1C_1C_2\mathbb{J}N_2$), then no C_i can receive all the correct answers because answers from $\mathbb{M}$ and $\mathbb{J}$ can't pass through N_1 and N_2. Thus the string of C_i must be continuous and bordered by $\mathbb{M}$ and $\mathbb{J}$.

Thus, it suffices to count all strings that contain a continuous block of all C_i bordered by $\mathbb{M}$ and $\mathbb{J}$. Counting this is reasonably straightforward: There are 3! ways to arrange the C_i, 2 ways to place $\mathbb{M}$ and $\mathbb{J}$ around the block ($\mathbb{M}$ first or $\mathbb{J}$ first), and 3! ways to arrange the block and the N_j. So, the probability all cheaters get a perfect score is

$$\frac{3!\cdot 2\cdot 3!}{7!} = \frac{6\cdot 2}{7\cdot 6\cdot 5\cdot 4} = \frac{1}{7\cdot 5\cdot 2}.$$

We conclude that the probability that all cheaters get a perfect score is sufficiently small: $\boxed{\frac{1}{70}}$. Don't cheat.

Remark: While the rigor in the solution is annoying, the idea is not that complex. □

Proposed by Nir Elber.