2020 AMC Mathcounts Solutions Packet

Austin Math Circle

To be used Jan. 2020

Sprint Round

Problem 1. Marian is entering the long jump event at the Olympics. If his speed in the air is always a constant 6 feet per second, and he spends 3.5 seconds in the air, how many **yards** does he travel during the time he jumps to the time he lands?

Solution. He jumps 2 yards per second, and as he spends 3.5 seconds in the air, he jumps $\boxed{7}$ yards. □

Proposed by Josiah Kiok.

Problem 2. If $a \spadesuit b = \frac{a+b}{ab+2}$, compute $20 \spadesuit 20$. Express your answer as a common fraction.

Solution. We write $20 \spadesuit 20 = \frac{20+20}{20 \cdot 20+2} = \frac{40}{402} = \boxed{\frac{20}{201}}$. □

Proposed by Matthew Kroesche.

Problem 3. Compute 1111×2222.

Solution. We write this as

$$2222 + 22220 + 222200 + 2222000$$

We can just add this through without any carrying to get $\boxed{2468642}$. □

Proposed by Matthew Kroesche.

Problem 4. The base-three number 2020_3 is equal to what number in base ten?

Solution. We write this as

$$2 \cdot 3^3 + 0 \cdot 3^2 + 2 \cdot 3 + 0 \cdot 1 = 54 + 6 = \boxed{60}$$

□

Proposed by Matthew Kroesche.

Problem 5. Annie is placing either a blue circle or a purple star into each square of a 3 by 3 tic-tac-toe grid with one restriction: No two blue circles may be placed in adjacent squares. In how many ways can this be done if there is a blue circle in the middle square?

Solution. Since there is a blue circle in the middle square, the four squares adjacent to it must contain purple stars. This leaves only the corner squares to be colored, and each corner square may contain either a blue circle or a purple star since none of the corner squares are adjacent to each other or the blue circle in the middle square. Our answer is 2^4 which is equal to $\boxed{16}$. □

Proposed by Saskia Solotko.

Problem 6. Bryan glues 27 wooden blocks in the shape of unit cubes together to form a large $3 \times 3 \times 3$ cube. He then removes the 8 blocks that are located at the 8 vertices of the larger cube, so that only 19 blocks remain. What is the surface area of the resulting solid?

Solution. Removing a block in this manner does not change the surface area, since the block had a net surface area of 6, 3 square units of which were outside the larger cube, and 3 of which were inside it. Thus the surface area after Bryan removes the eight blocks is the same as the surface area before he removed them, which is $6 \times 9 = \boxed{54}$. □

Proposed by Matthew Kroesche.

Problem 7. Compute the area of the triangle bounded by the x-axis, the y-axis, and the line $4x+5y=6$. Express your answer as a common fraction.

Solution. The x-intercept is at $x=\frac{6}{4}=\frac{3}{2}$, and the y-intercept is at $y=\frac{6}{5}$. Thus the area is $\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{6}{5}=\boxed{\frac{9}{10}}$. $\square$

Proposed by Matthew Kroesche.

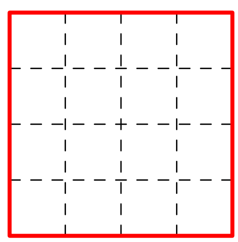
Problem 8. Compute the number of ways to arrange 3 identical checkers on a 3×3 checkerboard given that they do not all lie on the same row or column.

Solution. We do complementary counting. There are, in total, $\binom{9}{3}$ ways to choose 3 spots for our checkers out of the total 9. However, there are 3 rows and 3 columns which must be filled full, which accounts for 6 ways we must subtract.

Our final answer is $\frac{9\cdot 8\cdot 7}{3\cdot 2\cdot 1}-6=3\cdot(4\cdot 7-2)=3\cdot 26=\boxed{78}$. $\square$

Proposed by Nir Elber.

Problem 9. Bob paints the edges of a 4×4 square and then cuts the 4×4 vertically and horizontally to make sixteen 1×1 squares, as shown. He then chooses a random 1×1 square. What is the expected number of painted edges this square has?



Solution 1. Because there are $4\cdot 4=16$ total painted edges and $4\cdot 16=64$ total edges, the probability that any one randomly chosen edge is painted is equal to

$$\frac{16}{64}=\frac{1}{4}.$$

By linearity of expectation, because a square has 4 painted edges, the final answer is $4\cdot\frac{1}{4}=\boxed{1}$. $\square$

Solution 2. An easier-to-generate solution follows. We proceed with casework.

The probability that the number of painted edges is 0 is equal to the number of squares with 0 painted edges divided by the total. This is $\frac{4}{16}$. Similarly, the probability that the number of painted edges is 1 is $\frac{8}{16}$, and the probability that the number of painted edges is 2 is $\frac{4}{16}$. Note a square cannot have more than 3 painted edges.

Thus, our answer is

$$0\cdot\frac{4}{16}+1\cdot\frac{8}{16}+2\cdot\frac{4}{16}=\frac{16}{16}=\boxed{1}.$$

Notice this matches the answer from the previous solution. $\square$

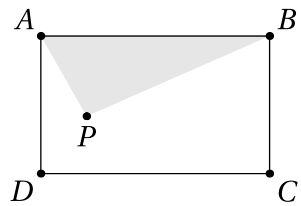
Proposed by Nir Elber.

Problem 10. Find the smallest number divisible by 2^3 with exactly three 2's as digits.

Solution. Quick casework solves the problem sufficiently quickly. Note that three digits does not make sense. For four digits we move on to _222, 2_22, 22_2 and 222_.The first and second have no solution, the smallest number for the third case is 2232, but the fourth returns $\boxed{2224}$ as the correct answer. $\square$

Proposed by Ethan Liu.

Problem 11. Rectangle $ABCD$ has $AB = CD = 5$ and $AD = BC = 3$. A random point P is chosen from its interior. Compute the expected value of the area of $\triangle APB$. Express your answer as a common fraction.



Solution. For some P, let the distance from P to $\overline{AB}$ be denoted h, and notice that it is evenly distributed between 0 (P on $\overline{AB}$) and 3 (P on $\overline{CD}$) with no favoritism. It follows that the expected value of h is $\frac{3}{2}$.

Thus, the expected value of the area is $\frac{1}{2} \cdot AB \cdot h = \boxed{\frac{15}{4}}$. □

Proposed by Nir Elber.

Problem 12. Anna and Bob start with particular integer amounts of dollars. Anna has three less dollars than Bob at the start. First, Anna gives exactly half of her dollars to Bob. Then, Bob donates exactly two-thirds of his dollars to charity. How many more dollars does Bob have than Anna?

Solution. Let Anna have x dollars, and Bob have $x+3$ dollars. When Anna gives half her dollars to Bob, she will have $\frac{x}{2}$ dollars and Bob will have $\frac{3x}{2}+3$ dollars. After Bob donates his money to charity, he will have $\frac{x}{2}+1$ dollar, and Anna still has $\frac{x}{2}$. Therefore, the answer is $\boxed{1}$. □

Proposed by Josiah Kiok.

Problem 13. Jehu and Heyu each randomly think of an integer from 1 to 9 inclusive. What is the probability that the product of their integers is even? Express your answer as a common fraction.

Solution. The probability that Jehu's integer is odd is $\frac{5}{9}$, and the probability that Heyu's integer is odd is $\frac{5}{9}$ as well. Thus, multiplying together (since the result will only be odd if both of their individual numbers were odd), the probability that the product is odd is $\frac{25}{81}$. Thus the probability that it's even is $1-\frac{25}{81} = \boxed{\dfrac{56}{81}}$. □

Remark: The names in this problem are pronounced HAY-oo and HI-yoo respectively.

Proposed by Matthew Kroesche.

Problem 14. How many distinct positive integers divide the number 91091?

Solution. We see that $91 = 7 \cdot 13$, and that $91091 = 91 \cdot 1001 = 7 \cdot 13 \cdot 7 \cdot 11 \cdot 13 = 7^2 \cdot 11 \cdot 13^2$. Thus, the number of positive integer divisors is $3 \cdot 2 \cdot 3 = \boxed{18}$. □

Proposed by Matthew Kroesche.

Problem 15. David has a lot of chocolate bars. He gives seven of his chocolate bars to Lee. Then, he gives a third of his remaining chocolate bars to Ben. Finally, he eats five of his chocolate bars. Afterwards, he discovers that he has precisely half as many chocolate bars as he originally had. How many chocolate bars did David originally have?

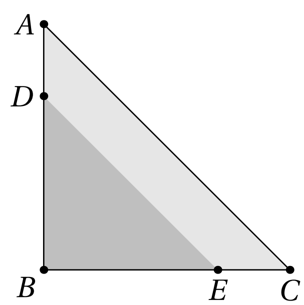
Solution. Call this number n. Then after David gives seven chocolate bars to Lee, he has $n-7$. After he gives a third of them to Ben, he has $\frac{2}{3}(n-7) = \frac{2n-14}{3}$ remaining. Finally, after he eats five more, he has $\frac{2n-14}{3} - 5 = \frac{2n-29}{3}$. Setting this number equal to $\frac{n}{2}$, we have

$$\frac{2n-29}{3} = \frac{n}{2}$$
$$4n - 58 = 3n$$
$$n = \boxed{58}$$

□

Proposed by Matthew Kroesche.

Problem 16. In right isosceles triangle $\triangle ABC$ with right angle B, points D and E are chosen on $\overline{AB}$ and $\overline{BC}$ respectively so that $\overline{DE}$ is parallel to $\overline{AC}$ and that $\triangle DBE$ has equal to area to quadrilateral $ADEC$. Given $AB = 2$, compute DE.



Solution. We begin by restating the area condition as

$$2[\triangle DBE] = [\triangle DBE] + [\triangle DBE] + [\triangle DBE] + [ADEC] = [\triangle ABC].$$

Now, the key observation is that $\triangle DBE \sim \triangle ABC$. Indeed, $\angle BDE = \angle BAC$ and $\angle BED = \angle BCA$ due to corresponding angles from $\overline{DE} \parallel \overline{AC}$. It follows that

$$\left(\frac{DE}{AC}\right)^2 = \frac{[\triangle DBE]}{[\triangle ABC]} = \frac{1}{2}$$

because the ratio of the areas is the square of the ratio of the sides for similar figures. However, from the Pythagorean theorem, we have that

$$AC^2 = AB^2 + BC^2 = 2^2 + 2^2 = 8$$

because $\triangle ABC$ is right isoceles, so we can now finish the problem. Note

$$DE^2 = \frac{1}{2}AC^2 = \frac{1}{2} \cdot 8 = 4,$$

so we have $DE = \boxed{2}$. □

Proposed by Nir Elber.

Problem 17. Nathan chooses a positive integer n whose largest divisor, other than n, is 2020. Compute the sum of all possible values of n.

Solution. Define $m > 0$ so that $n = 2020m$. Note $m \neq 1$ because this violates the condition that 2020 is a proper divisor. But if $m \neq 2$, then $m > 2$, so

$$n = 2020m = 2 \cdot (1010m),$$

and $1010m > 1010 \cdot 2 = 2020$ is a larger proper divisor of n. Thus, we must have $m = 2$, so $n = 2020 \cdot 2 = \boxed{4040}$. □

Proposed by Nir Elber.

Problem 18. Square $MACK$ has side length 3. A line segment, connecting segment $\overline{MA}$ to segment $\overline{CK}$ and parallel to segment $\overline{AC}$, divides square $MACK$ into two rectangles, such that the perimeter of one is numerically equal to the area of the other. Find this common value, expressed as a common fraction.

Solution. Suppose the rectangle whose perimeter is numerically equal to the area of the other rectangle has its shorter side of length t. Then its perimeter is $3+t+3+t=6+2t$, and the area of the other rectangle is $3(3-t)=9-3t$. We thus set $6+2t=9-3t$ to get $5t=3 \implies t=\frac{3}{5}$. Then the common value is $6+2\cdot\frac{3}{5}=\boxed{\frac{36}{5}}$. □

Proposed by Matthew Kroesche.

Problem 19. Suppose each of the letters A, B, N represents a nonzero digit, such that the sum of the four-digit number $ANNA$ and the six-digit number $BANANA$ is 996656. Compute the sum $A+B+N$.

Solution. We see that the units digit of $A+A$ is 6, so A is either 3 or 8. If A were 3, there would be no carry, so the tens digit would be the units digit of $N+N$ which must be even. Since the tens digit is 5, it must be then that $A=8$, and that a 1 is carried out so that 5 is the units digit of $2N+1$. Then N is either 2 or 7. If N were 2, there would be no carry, and then the next column would be $A+N=8+2$ which has a units digit of 0. Since the hundreds digit is 6, it must be the other one. Indeed, if $N=7$, then the hundreds column becomes $8+7+1=16$ due to the 1 being carried. Then a 1 is carried into the thousands column, so it is also $8+7+1=16$. For the next column to the left, it works out to $1+A=9$ with no carry, so then we have $B=9$. Thus $A+B+N=8+9+7=\boxed{24}$ □

Proposed by Matthew Kroesche.

Problem 20. Compute the least positive integer n such that

$$\sqrt{1+\sqrt{2+\sqrt{3+\sqrt{4+n}}}}$$

is an integer.

Solution. We see that this expression is increasing as n increases, and furthermore that whenever it is an integer, n will also be an integer since we can repeatedly square both sides and subtract. It can't be equal to 1, since $\sqrt{1+\sqrt{2+\sqrt{3}}}>\sqrt{1}=1$, so we set it equal to 2, the next smallest possible positive integer. This gives

$$\sqrt{1+\sqrt{2+\sqrt{3+\sqrt{4+n}}}}=2$$

$$1+\sqrt{2+\sqrt{3+\sqrt{4+n}}}=4$$

$$\sqrt{2+\sqrt{3+\sqrt{4+n}}}=3$$

$$2+\sqrt{3+\sqrt{4+n}}=9$$

$$\sqrt{3+\sqrt{4+n}}=7$$

$$3+\sqrt{4+n}=49$$

$$\sqrt{4+n}=46$$

$$4+n=2116$$

$$n=\boxed{2112}$$

□

Proposed by Matthew Kroesche.

Problem 21. Alex randomly writes down *nonzero* digits from left to right until the concatenated digits form a number greater than 500. For example, he might write 1, 15, 159, ending at 1594. Compute the expected number of digits Alex writes. Express your answer as a common fraction.

Solution. Notice that Alex writes either 3 or 4 digits. If the first digit is less than 5, then Alex will have to write 4 digits to end up greater than 500, an conversely, if the first digit is greater than or equal to 5, then Alex will only have to three digits. Observe that Alex never writes 0, so 001 or even 500 are all not achievable. Thus, it remains to compute

$$\underbrace{\frac{4}{9}}_{\text{first digit}<5} \cdot \underbrace{4}_{\text{digits}} + \underbrace{\frac{5}{9}}_{\text{first digit}\geq 5} \cdot \underbrace{3}_{\text{digits}}.$$

This is $\frac{16+15}{9} = \boxed{\frac{31}{9}}$. □

Proposed by Nir Elber.

Problem 22. Let $a \odot b = a - \frac{a}{b}$ for $b \neq 0$. Compute $\Big(\big((6 \odot 5) \odot 4\big) \odot 3\Big) \odot 2$. Express your answer as a common fraction.

Solution. The trick is to factor it so that it telescopes. Note

$$a \odot b = a - \frac{a}{b} = a\left(1 - \frac{1}{b}\right) = a \cdot \frac{b-1}{b}.$$

Thus, the desired expression is just

$$\Big(\big((6 \odot 5) \odot 4\big) \odot 3\Big) \odot 2 = 6 \cdot \frac{4}{5} \cdot \frac{3}{4} \cdot \frac{2}{3} \cdot \frac{1}{2} = \boxed{\frac{6}{5}}.$$

So we are done. □

Proposed by Nir Elber.

Problem 23. Let a_n be a sequence where $a_0 = 2$, $a_1 = 5$, and for each $n > 2$, let a_n be the remainder when ${a_{n-2}}^{a_{n-1}}$ is divided by 7. Find a_{2020}.

Solution. We search for a pattern.
$a_2 = 2^5 \pmod 7 = 4.$
$a_3 = 5^4 \pmod 7 = 2.$
$a_4 = 4^2 \pmod 7 = 2.$
$a_5 = 2^2 \pmod 7 = 4.$
Since for any number $n \cong 1 \pmod 3$ greater than 1 $a_n = 2$, $a_{2020} = \boxed{2}$ as well. □

Proposed by Ethan Liu.

Problem 24. Find the sum of all possible values of x that satisfies $x^2 + y^2 + 4x + 4y = 122$, where x and y are positive integers with $x < y$.

Solution. Completing the square for both x and y gives $(x+2)^2 + (y+2)^2 = 130$. The two perfect squares that sum up to 130 can be found with normal searching methods to be either 9 and 121, or 49 and 81. The first case gives $x = 1$, and the second case gives $x = 5$, for a total answer of $\boxed{6}$. □

Proposed by Josiah Kiok.

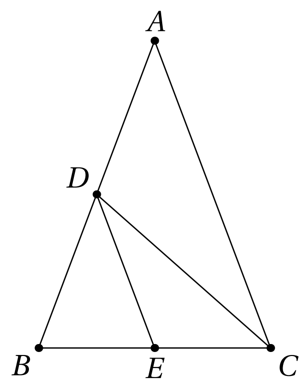
Problem 25. Triangle ABC is isosceles with $AB = AC$. Point D lies on side $\overline{AB}$ so that $BC = CD$, and point E lies on side $\overline{BC}$ so that $BD = DE$. If $BE = CE = 1$, compute AE. Express your answer in simplest radical form.

Solution. The key here is to realize that $ABC \sim CDB \sim DBE$, since they are all isosceles and share the common base angle $\angle B$. Then if $BE = CE = 1$, let $BD = DE = x$. Then $CD = 2$, and thus due to similarity,

$$\frac{CD}{DB} = \frac{DB}{BE}$$

and thus $\frac{2}{x} = \frac{x}{1}$ so $x = \sqrt{2}$. Furthermore, since ABC is also similar to these two triangles, we must have $AB = AC = \sqrt{2} \cdot BC = 2\sqrt{2}$. Since E is the midpoint of BC and ABC is isosceles, we also have $AE \perp BC$, so by the Pythagorean Theorem

$$AE = \sqrt{AB^2 - BE^2} = \sqrt{8-1} = \boxed{\sqrt{7}}$$



□

Proposed by Matthew Kroesche.

Problem 26. Let $f(n) = n + 2\sqrt{n-1} + 1$. Compute $f(f(f(f(f(f(f(f(f(f(10))))))))))$. (The function f is applied 10 times.)

Solution. One possibility is just to hammer out the values, but this wastes a ton of time. A much faster solution is to see that $f(n) = (\sqrt{n-1}+1)^2+1$. Thus $f(n^2+1) = (n+1)^2+1$, so we see that $f(f(f(f(f(f(f(f(f(f(3^2+1)))))))))) = 13^2 + 1 = \boxed{170}$. □

Proposed by Matthew Kroesche.

Problem 27. What is the 2020th positive integer with only even digits?

Solution. When the list of all positive integers with even digits is divided by two, the result is a the list of numbers with digits with all digits less than 5. Therefore, the new list is equivalent (should I say isomorphic?) to a list of base 5 numbers. So, since $2020_{10} = 31040_5$, the answer is $\boxed{62080}$. □

Proposed by Joshua Pate.

Problem 28. A bag contains 3 red marbles, 10 blue marbles, and 13 green marbles. Ten marbles are drawn from the bag, without replacement. Given that the probability the next two marbles drawn are green is $\frac{3}{10}$, compute the number of green marbles in the first ten drawn.

Solution. Suppose that exactly $13-g$ out of the first ten marbles drawn were green so that there are currently g green marbles; then because there is currently a total of $3+10+13-10=16$ marbles left over, the probability that the next two marbles are green is

$$\frac{g}{16}\cdot\frac{g-1}{15}=\frac{3}{10}.$$

Clearing fractions, we see that

$$g(g-1)=72,$$

so we are searching for consecutive factors of 72. By inspection, the numbers are approximately $\sqrt{72}\in[8,9]$, and testing $g=9$ reveals that it works.

It follows that $13-g=\boxed{4}$ of the marbles in the first ten were green. □

Proposed by Nir Elber.

Problem 29. Let $T(n)$ be the number of trailing 0s of n when n is written in base 3. For example, $T(4_{10})=T(11_3)=0$ and $T(18_{10})=T(200_3)=2$. Compute $T(1)-T(2)+T(3)-\ldots+T(243)$.

Solution 1. Observe that the number of trailing 0s of k is simply the number of factors of 3 k has; we denote this $v_3(k)$. Thus, the problem is asking us to compute

$$S=\sum_{k=1}^{3^5}(-1)^{k+1}v_3(k).$$

The key idea in this solution is that for k between 1 and 3^5-1 inclusive, the number of trailing 0s of k is the same as the number of trailing 0s as 3^5-k. Informally, we can consider the subtraction and note that the leftmost nonzero digit of k stays the leftmost nonzero digit of 3^5-k. Formally, note that for ℓ between 1 and 5 inclusive, $3^\ell\mid k$ if and only if $3^\ell\mid 3^5-k$. Because $v_3(k)<5$, the result follows.

Now we use this to pair off first and last terms. In full, the algebra is as follows.

$$\begin{aligned}
S&=\underbrace{\left(\sum_{k=1}^{(3^5-1)/2}(-1)^{k+1}v_3(k)\right)}_{S'}+\left(\sum_{k=(3^5+1)/2}^{3^5-1}(-1)^{k+1}v_3(k)\right)+(-1)^{3^5+1}v_3\left(3^5\right)\\
&=S'+\left(\sum_{k=1}^{(3^5-1)/2}(-1)^{3^5-k+1}v_3\left(3^5-k\right)\right)+(-1)^{3^5+1}v_3\left(3^5\right)\\
&=S'+(-1)^{3^5-2k}\left(\sum_{k=1}^{(3^5-1)/2}(-1)^{k+1}v_3(k)\right)+(-1)^{3^5+1}v_3\left(3^5\right)\\
&=S'-S'+(-1)^{3^5+1}v_3\left(3^5\right)\\
&=\boxed{5},
\end{aligned}$$

so we are done. □

Solution 2. Once again, we are summing $(-1)^{k+1}v_3(k)$ for k between 1 and 3^5 inclusive. Taking a hint from Legendre, instead of counting this over k, we count this over individual powers of 3.

The integers whose last base-3 digit is a 0 are $\left\{3\cdot1, 3\cdot2,\ldots,3\cdot3^4\right\}$. Each of these contribute their last digit to the total as a $+1$ or -1 to the sum depending on parity. However, pairing an odd $3(2k-1)$ in the set with its neighbor $3(2k)$, we see that there is exactly one more odd than even (3^5 has no neighbor), so in total this case contributes $+1$ to the total.

Similarly, the integers whose last 2 base-3 digits are a 0 are $\left\{3^2\cdot1,3^2\cdot2,\ldots,3^2\cdot3^3\right\}$. Because last digits have already been counted, we count the second-to-last digit, which again contribute $+1$ or -1. Doing the same pairing, all odds except for 3^5 have an even partner to their left, so this contributes a total of $+1$ to the total.

This logic continues. The integers whose last 3 base-3 digits are all 0 are $\left\{3^3\cdot1,\ldots,3^3\cdot3^3\right\}$, and for each third-to-last 0, each $+1$ given by an odd element can be paired with a -1 from an even element to its left with the exception of 3^5 which has no neighbor. Thus, $+1$ gets added to the total.

Further, the integers whose last 4 base-3 digits are all 0 are $\{3^4 \cdot 1, 3^4 \cdot 2, 3^4 \cdot 3\}$, which sums to $+1 - 1 + 1 = +1$ when counting just the fourth-to-last 0.

Finally, 3^5 is the only integer with 5 trailing 0s, and as 4 of them have already been accounted for, we only append another +1 to the total.

Thus, Theodore's final total is $1 + 1 + 1 + 1 + 1 = \boxed{5}$. □

Proposed by Nir Elber.

Problem 30. Quadrilateral $ABCD$ is inscribed in a circle. Let E be the intersection of segments $\overline{AC}$ and $\overline{BD}$, and let F be the intersection of lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$. If $BE = 2, AE = 5$. Find $\frac{CF}{FA}$. Express your answer as a common fraction.

Solution. Let G be the point on $\overline{AC}$ such that $AF = FG$. By angle chasing $\Delta GFC \sim \Delta AEB$. Thus, $\frac{BE}{AE} = \frac{CF}{FG} = \frac{CF}{FA}$, so the answer is obviously $\boxed{\frac{2}{5}}$. □

Proposed by Ethan Liu.

Target Round

Problem 1. A restaurant sells cups of boba tea in 7-ounce and 12-ounce quantities. Bill wants to buy some number of cups of tea totaling exactly 100 ounces. How many of the 7-ounce cups should he buy?

Solution. We see by inspection that

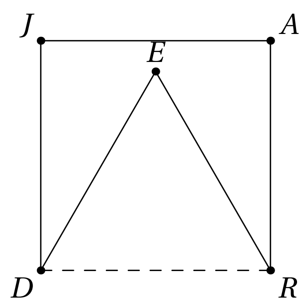
$$7 \cdot 4 + 12 \cdot 6 = 100$$

so the answer is $\boxed{4}$. □

Proposed by Matthew Kroesche.

Problem 2. In concave pentagon $JARED$, $\angle J = \angle A = 90°$, $\angle R = \angle D = 30°$, $\angle E = 300°$, and all five sides have length 2. Compute the area of the pentagon. Express your answer in simplest radical form.

Solution. Drawing the picture reveals that the pentagon is a square with an equilateral triangle removed from its interior:



Thus the area is $\boxed{4-\sqrt{3}}$ □

Proposed by Matthew Kroesche.

Problem 3. Given $f(x) + f(y) = f(xy)$ and $f(x) \neq 0$ for any integer x, find $\frac{f(1024)}{f(8)}$. Express your answer as a common fraction.

Solution. We know that $f(2)+f(2)+f(2) = f(8)$ and $f(2)+f(2)+f(2)+f(2)+f(2)+f(2)+f(2)+f(2)+f(2)+f(2) = f(1024)$. Knowing $f(2)$ is nonzero, the answer is $\boxed{\frac{10}{3}}$ □

Proposed by Ethan Liu.

Problem 4. Mistaken Melinda keeps making off-by-one errors! She thought earlier today that the remainder when some positive integer $n \leq 35$ was divided by 2, 3, 5, and 7 were all 1, but she now realized that each of her remainders could be off by one (plus or minus) or not off at all. Compute the number of possible n.

Solution. This problem is not as exciting as it appears. To deal with the low-hanging fruit first, observe that being off by 1 when divided by 2 or 3 gives literally no information: The only residues $\pmod 2$ are $\{0,1\}$, and the only residues $\pmod 3$ are $\{0,1,2\}$, so everything is 1 (or 0) away from 1.

So now we have to do more work. Because the information about 2 and 3 is useless, we focus on the 5 and the 7. Essentially, we are given that

$$n \equiv \begin{cases} 0,1,2 & \pmod 5 \\ 0,1,2 & \pmod 7 \end{cases}.$$

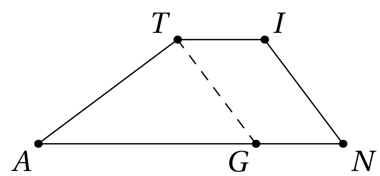
Imagine choosing to integers $a, b \in \{0, 1, 2\}$ to correspond to the residues modulus 5 and 7 respectively. By the Chinese Remainder Theorem, each ordered pair (a, b) corresponds to one unique integer $\pmod{35}$, so there will be 1 solution for $n \in [1, 35]$.

Because there are $3 \cdot 3 = 9$ such choices for (a, b), and 1 choice for n for each (a, b), we conclude that there are a total of $\boxed{9}$ total solutions for n. □

Proposed by Nir Elber.

Problem 5. Trapezoid $TINA$ has $TI \parallel NA$, $TI = 2$, $IN = 3$, $NA = 7$, $AT = 4$. Compute the area of trapezoid $TINA$. Express your answer as a common fraction.

Solution. Let G be the point on segment NA that is two units away from N. Then $TING$ is a parallelogram, and so $IN = GT = 3$. Furthermore $AT = 4$ and $GA = 5$, so TAG is a $3-4-5$ right triangle. The height of $TINA$ is thus equal to the altitude to the hypotenuse of TAG, which is $\frac{12}{5}$. Thus the area of $TINA$ is $\frac{12}{5}\left(\frac{7+2}{2}\right) = \boxed{\frac{54}{5}}$.



□

Proposed by Matthew Kroesche.

Problem 6. Mews the cat and Yertle the turtle are running a race. They run at constant speeds, with Mews running six times as fast as Yertle. To compensate for this, Yertle is allowed to start fifty feet ahead of the starting line, while Mews starts at the starting line. When the start signal is given, they both begin running. Fifteen seconds after the beginning of the race, Mews passes Yertle. However, once Mews reaches the finish line, she turns around and begins running back towards the start at the same speed, and she passes Yertle again fifty-four seconds after the beginning of the race. How many feet long is the race?

Solution. Let Yertle run at a speed of r (in feet per second), so that Mews runs at a speed of $6r$, and let the length of the race be d feet. Then Yertle's position at time t is $rt + 50$, and Mews's position at time t is $6rt$ before she turns around, and $2d - 6rt$ after she turns around. Plugging in $t = 15$ gives $15r + 50 = 90r \implies r = \frac{2}{3}$. Since the second time the two meet is at $t = 15 + 39 = 54$, we also have that $\frac{2}{3} \cdot 54 + 50 = 2d - 6 \cdot \frac{2}{3} \cdot 54 \implies 2d = 36 + 50 + 216 = 302$, and so $d = \boxed{151}$ feet. □

Proposed by Matthew Kroesche.

Problem 7. What is the probability that a random arrangement of the letters in the word AMERICA will contain the word ERICA as a substring? Express your answer as a common fraction.

Solution. There are $\frac{7!}{2}$ permutations of these letters, where we divided by 2 because there are two A's. There are 6 permutations that contain ERICA as a substring: three choices for whether the E is the first, second, or third letter of the word, and two choices for what order the A and M we didn't use should go in. Thus the answer is

$$\frac{6}{7!/2} = \frac{1}{7 \cdot 5!/2} = \frac{1}{7 \cdot 60} = \boxed{\frac{1}{420}}$$

□

Remark: I resisted the urge to ask instead for the probability that the letters E,R,I,C,A appear in order but are not necessarily consecutive; this would make the problem much harder. (The answer becomes $\frac{1}{72}$ in this case.)

Proposed by Matthew Kroesche.

Problem 8. An arrow starts at $(0,0)$ pointing upwards. Every second it either turns clockwise 90 degrees or moves one unit in the direction it is pointing. Find its expected location after 6 seconds. Express your answer as an ordered pair (x, y) where each of x, y is a common fraction.

Solution. We use recursion. Say the expected point at i seconds is at E_i which is the average destination of all paths. Since one can insert a 90 degree rotation or a 1 unit translation at the beginning, E_{i+1} is the average of E_i rotated 90 degrees and E_i translated upwards by 1 which is $\frac{x_i+y_i}{2}, \frac{y_i+1-x_i}{2}$ where x_i, y_i are the x and y coordinates of E_i respectively. Thus,

$E_1 = (0, \frac{1}{2})$
$E_2 = (\frac{1}{4}, \frac{3}{4})$
$E_3 = (\frac{1}{2}, \frac{3}{4})$
$E_4 = (\frac{5}{8}, \frac{5}{8})$
$E_5 = (\frac{5}{8}, \frac{1}{2})$

$$\boxed{E_6 = (\frac{9}{16}, \frac{7}{16})}$$

□

Proposed by Ethan Liu.

Team Round

Problem 1. Steven has a solid cube of side length 1. He glues six right square pyramids, each with base side length 1 and altitude $\frac{1}{2}$, to the six faces of his cube. How many faces does the resulting solid have?

Solution. While one might instinctively think there are $6 \times 4 = 24$ faces, in fact for every edge of the cube, the two triangular faces that share that edge lie in the same plane, so each pair of triangular faces in fact is replaced with one rhombus. As a result, the solid actually has only $\boxed{12}$ faces. □

Proposed by Matthew Kroesche.

Problem 2. Find the number of 4-person committees that can be formed out of 7 boys and 8 girls such that the committee has at least one person of each gender.

Solution. Since there are $\binom{15}{4}$ total possible committees, $\binom{7}{4}$ all-boy committees and $\binom{8}{4}$ all-girl committees, the answer is $\binom{15}{4} - \binom{7}{4} - \binom{8}{4} = \boxed{1260}$

Solution 2.

We do casework on the number of boys.

Case 1: 1 boy, 3 girls. $\binom{7}{1}\binom{8}{3}$

Case 2: 2 boys, 2 girls. $\binom{7}{2}\binom{8}{2}$

Case 3: 3 boys, 1 girl. $\binom{7}{3}\binom{8}{1}$

Sum them and we obtain $\boxed{1260}$

□

Remark: Combine the solutions and we obtain a special case of Vandermonde's Identity.

Proposed by Ethan Liu.

Problem 3. The height of Hubris is 6 meters, and the height of Arrogance is 2 meters. Each year, Hubris gains half the height of Arrogance (at the start of the year), and Arrogance gains half the height of Hubris (at the start of the year). After four years, what is the total height of Hubris and Arrogance? Express your answer as a common fraction.

Solution. Let $H_0 = 6$ and $A_0 = 2$ so that we define H_n and A_n to be the heights of Hubris and Arogance, respectively, after n years. By the problem statement, we have the following system of linear recurrences:

$$\begin{cases} H_{n+1} = H_n + \frac{1}{2}A_n \\ A_{n+1} = A_n + \frac{1}{2}H_n \end{cases}.$$

The key step is not to evaluate H_n and A_n individually (although this is doable) but rather to sum them:

$$H_{n+1} + A_{n+1} = H_n + \frac{1}{2}A_n + A_n + \frac{1}{2}H_n = \frac{3}{2}H_n + \frac{3}{2}A_n = \frac{3}{2}(H_n + A_n).$$

It follows that the sequence $H_n + A_n$ is geometric with common ratio $\frac{3}{2}$, so after four years of iterating, this is

$$H_4 + A_4 = \left(\frac{3}{2}\right)^4 (H_0 + A_0) = \frac{81}{16} \cdot 8 = \boxed{\frac{81}{2}}.$$

Thus we are done. □

Remark: The numbers are chosen to make bashing a not terrible alternative to cleverness.

Proposed by Nir Elber.

Problem 4. Two finalists are given a set of four true-false questions, and they both guess randomly. A question is called *spicy* if nobody guessed it correctly. Compute the probability there was a spicy question. Express your answer as a common fraction.

Solution. Without loss of generality, all of the answers are false. For a spicy question to occur, all students would have to guess true on that question, which occurs with probability $\frac{1}{2^2}$. Thus, the probability of a particular question not being spicy is $1-\frac{1}{2^2}$. Joining this will all questions, the probability that no questions were spicy is

$$\left(1-\frac{1}{2^2}\right)^4 = \left(\frac{3}{4}\right)^4 = \frac{81}{256}.$$

Finally, the probability that there was a spicy question is $1-\frac{81}{256} = \boxed{\frac{175}{256}}$. □

Proposed by Nir Elber.

Problem 5. Compute the number of zeros at the end of the number

$$1^1 \cdot 2^2 \cdot 3^3 \cdots 29^{29} \cdot 30^{30}$$

Solution. We find the number of powers of five that divide the number – the number of twos can be easily seen to be much greater. Thus we see, after making sure to count each five from the 25^{25} twice, that the answer is $5+10+15+20+25+25+30 = \boxed{130}$. (We can see that, even with the 25^{25} in the expression, there are still more twos than fives, e.g., 24^{24} has more twos than 25^{25} has fives.) □

Proposed by Matthew Kroesche.

Problem 6. Compute the number of ordered triplets of positive integers (a,b,c) such that $abc = 10!$ and no two of a, b, c share a common factor larger than 1.

Solution. Observe that if a prime $p \mid a \mid 10!$, then that p cannot divide into b or c, lest that contribute to the common factor between two of those numbers. So because all factors of p that go into 10! cannot be split among two elements, they must all go into one element, in this case a. It follows that what we are really doing is counting the number of ways to partition the set of prime factors of 10! into our three positive integers.

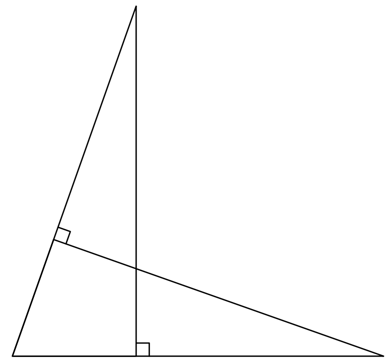
The only prime factors of 10! are 2, 3, 5, 7, and each of these have three options: Either the 2s go into a, go into b, or go into c; similar holds for the other primes. Thus, our final answer is

$$\underbrace{3}_{2} \cdot \underbrace{3}_{3} \cdot \underbrace{3}_{5} \cdot \underbrace{3}_{7} = \boxed{81}.$$

Done. □

Proposed by Nir Elber.

Problem 7. In the figure below, the line segments delineate three non-overlapping regions, each of which has area 1. What is the length of the segment marked x? Express your answer in simplest radical form.



Solution. We observe that all the right triangles in this diagram are similar, since they share a right angle and an acute angle. Furthermore, the two small triangles have the same area and are thus congruent, and the two big right triangles are congruent for the same reason. So since x is one of the legs of a small triangle, call the other leg length y. Then we know $\frac{1}{2}xy = 1$. We also know that the area of a big triangle is 2, so the legs of a big triangle must be $\sqrt{2}x$ and $\sqrt{2}y$. However, since the little triangle on top is congruent to the little triangle on the right, we have $\sqrt{x^2+y^2}+y = x\sqrt{2}$ due to calculating the length of the longer leg of a big triangle in two different ways. Subtracting y from both sides and squaring gives

$$x^2 + y^2 = 2x^2 - 2\sqrt{2}xy + y^2$$
$$x^2 = 2\sqrt{2}xy = 4\sqrt{2}$$

since $\frac{1}{2}xy = 1$. Then taking the square root, $x = \boxed{2\sqrt[4]{2}}$. □

Proposed by Matthew Kroesche.

Problem 8. Alex and Isabella take 2 minutes to finish one pint of ice cream, working together. Alex and Pierce take 3 minutes to finish one pint of ice cream, working together. Isabella and Pierce take 4 minutes to finish one pint of ice cream, working together. Alex buys three pints of ice cream. How long will it take all three of them to finish if Alex finishes one pint, passes it to Isabella when he's finished, who finishes one pint, who passes it to Pierce when she's finished, who then finishes the last pint? Round your answer to the nearest integer.

Solution. We set equations for the rates:

$$\frac{1}{A} + \frac{1}{I} = \frac{1}{2}$$
$$\frac{1}{A} + \frac{1}{P} = \frac{1}{3}$$
$$\frac{1}{I} + \frac{1}{P} = \frac{1}{4}$$

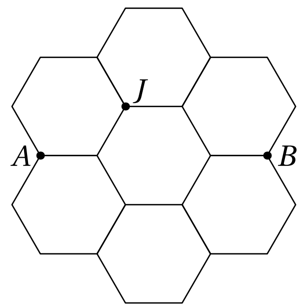
Seeing a symmetrical left hand side, we add all of the equations, giving $2\left(\frac{1}{A} + \frac{1}{I} + \frac{1}{P}\right) = \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$ This equation simplifies to $\frac{1}{A} + \frac{1}{I} + \frac{1}{P} = \frac{13}{24}$ Subtracting the first three equations from this and forming three new equations, we get:

$$\frac{1}{P} = 1/24 \text{ and therefore } P = 24$$
$$\frac{1}{I} = 5/24 \text{ and therefore } I = \frac{24}{5}$$
$$\frac{1}{A} = 7/24 \text{ and therefore } A = \frac{24}{7}$$

Our answer is then $24\left(1 + \frac{1}{5} + \frac{1}{7}\right) = \frac{47\cdot 24}{35} = \frac{1128}{35}$. This rounds to $\boxed{32}$. □

Proposed by Josiah Kiok.

Problem 9. Anson and Jason are playing a game. Anson has villages at the points labeled A and B on the grid below and wants to connect them via a road that follows the line segments on the grid and does not visit any point more than once. However, Jason has a village at the point marked J and so Anson's road may not pass through it. How many different roads could Anson build to accomplish this task?



Solution. There are two cases to consider. If Anson goes above Jason's village, then either he goes directly across the top of the board and then down to point B (there is only one way for him to do this) or he goes diagonally to the point just to the right of point J, and then has to follow the perimeter of the inner hexagon clockwise. There are three ways for him to get to point B by doing this, one for each line segment connecting the inner hexagon to the outer perimeter, since once he gets back to the outer perimeter, he has to go back counterclockwise to get to B. So we're at a total of 4 ways so far.

If Anson instead goes below Jason's village, consider the edges connecting the inner hexagon to the outer perimeter as he passes them going counterclockwise. There are 2 ways for him to reach a point on each edge for the first time from the previous edge, since he can travel around either the inside or the outside of the hexagon. Thus there are $2 \cdot 2 = 4$ ways for Anson to get to a point on the second edge around for the first time, and $2^3 = 8$ ways for him to get to a point on the third edge around for the first time. However, B is on the third edge and once he gets to B he is done (since he can't visit it more than once.) Thus, four of those eight ways for him to get to the third edge already have him ending at point B. For each of the other four, he goes to the point directly to the left of point B, and he can finish his journey by either going directly to the right, or by going over the top of the upper hexagon. Thus there are $4 \cdot 2 = 8$ paths of this form. Putting it all together, the total number of ways for Anson to get from A to B is $4 + 4 + 8 = \boxed{16}$. □

Remark: Yes, Anson and Jason are playing Catan.

Proposed by Matthew Kroesche.

Problem 10. Eight spheres have their centers at the vertices of a cube with side length 2, and each sphere has radius 1. For every set of four spheres whose centers form one face, there is a single sphere inside the cube tangent to all four spheres such that if two faces are neighboring, the corresponding spheres are tangent. Also, the centers of the internal spheres form a regular octahedron. Find the radius of one of these spheres. Express your answer in simplest radical form.

Solution. This would be good with a diagram, but good 3-D visualization is a simpler way to go. Finding the distance between the center of an internal sphere and its corresponding face we get $\sqrt{r^2 + 2r - 1}$, and finding the distance between the center of the same internal sphere and the center of the cube is $\sqrt{2}r$. These numbers sum to 1 so we obtain $\sqrt{r^2 + 2r - 1} + \sqrt{2}r = 1$. Solving, we obtain
$r^2 + 2r - 1 = 1 - 2\sqrt{2}r + 2r^2$
$r^2 - 2\left(\sqrt{2} + 1\right) r + 2$
$\left(r - \left(\sqrt{2} + 1\right)\right)^2 = 2\sqrt{2} + 1$. $r = \boxed{\sqrt{2} + 1 - \sqrt{2\sqrt{2} + 1}}$ □

Proposed by Ethan Liu.

Countdown Round

Problem 0. In the Welsh village of Llanfairpwllgwyngyllgogerychwyrndrobwllllantysiliogogogoch, a merchant is selling plums for the price of five firdlyc apiece. Llywelyn pays three ceiniogau for three plums, and receives a single firdlyc as change. Glyndwr pays six dymeu for two plums, and receives two firdlyc as change. Gwenwynwyn has twelve ceiniogau and sixteen dymeu. How many plums can he buy?

Solution. If Gwenwynwyn spends all twelve of his ceiniogau, he will be able to buy twelve plums and get four firdlyc as change. We see that six dymeu is worth two firdlyc more than two plums, which is a total of twelve firdlyc. Thus a dymey (singular) is worth two firdlyc, and 16 dymeu is thus worth 32 firdlyc. Adding to this the four firdlyc he got in change from buying the first twelve plums, Gwenwynwyn will have 36 firdlyc and thus be able to buy seven more plums with it. In total, then, Gwenwynwyn can buy $12 + 7 = \boxed{19}$ plums. □

Remark: These are all actual Welsh names. It would be very entertaining to see this as a Countdown problem, but not sure if it's a little too hard for that; it seems to be on the bubble.

Proposed by Matthew Kroesche.

Problem 1. Jay writes the numbers 1, 2, 3, and 27 on a blackboard, and Josh chooses two distinct integers from the blackboard. Compute the probability that the product of Josh's pair is a perfect cube. Express your answer as a common fraction.

Solution. Suppose Josh chooses an unordered pair $\{a, b\}$. Notice that if Josh chooses the 2 or the 3, then there is no possible way for the product to be a perfect cube. However, if Josh does not choose the 2 nor the 3 (i.e., the 1 and the 27), then the product is indeed a perfect cube. Thus, there is only 1 unordered pair whose product is a perfect cube.

Because there are a total of $\frac{4\cdot 3}{2} = 6$ total unordered pairs, the final probability is $\boxed{\frac{1}{6}}$. □

Proposed by Nir Elber.

Problem 2. Regular hexagon $AUSTIN$ and square $MATH$ lie in the same plane and share vertices A and T. If $AU = 2$, what is the area of the union of these two polygons? Express your answer in simplest radical form.

Solution. The area of any of the six equilateral triangles that make up hexagon $AUSTIN$ is $\sqrt{3}$. Furthermore, the side length of square $MATH$ is 4, so the area of $MATH$ is 16. Add to that the area of the three equilateral triangles in $AUSTIN$ that are not inside $MATH$, and the total area of the union is $\boxed{16 + 3\sqrt{3}}$. □

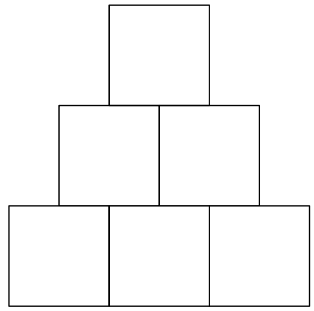
Proposed by Matthew Kroesche.

Problem 3. At 2:57 AM, Heffrey gets off of a bus. After buying 2 bags of candy \$1.59 each, he gets back on at 3:11 AM. How many minutes was Heffrey off of the bus?

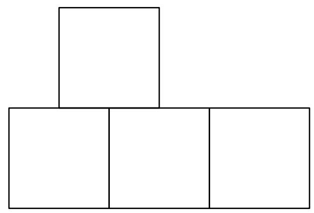
Solution. $\boxed{14}$ minutes have passed. □

Proposed by Ethan Liu.

Problem 4. Ashay has built a pyramid out of six blocks arranged as shown below. He wants to take the blocks away one at a time, but he cannot take a block away while there is another block resting on top of it. In how many ways can Ashay take away the six blocks?



Solution. First, Ashay must remove the top block. Then there are two choices for which block he removes next. After removing two blocks, Ashay will end up with either the following arrangement:



or a mirror image of it. From there, if Ashay removes the one block that's on top of the other two next, there are $3! = 6$ ways for him to finish since he can remove the last three blocks in any order. Otherwise, he has to remove the rightmost block, and then the top block, and then the remaining two in either order. Thus there are a total of $6 + 2 = 8$ ways for him to finish from this arrangement. Multiplying by the 2 from the beginning, the answer is $2 \cdot 8 = \boxed{16}$. □

Proposed by Matthew Kroesche.

Problem 5. Rick writes a two-digit integer on the board. Every minute, he replaces the number on the board with its square root, ending when the number is smaller than 2. Compute the largest number of square roots Rick performs.

Solution. Notice that making the integer larger cannot decrease the value of any of the square roots in sequence and thus cannot decrease the number of square roots taken. So it suffices to do the calculation for 99. Notice that

$$99 \xrightarrow{\sqrt{}} (9,10) \xrightarrow{\sqrt{}} (3,4) \xrightarrow{\sqrt{}} (1,2).$$

Thus, Rick must perform no more than $\boxed{3}$ square roots. □

Proposed by Nir Elber.

Problem 6. Compute the largest positive integer n such that 2^n divides

$$1! \times 2! \times 3! \times 4! \times 5! \times 6! \times 7! \times 8!$$

Solution. We compute the number of twos in each factor iteratively: the first factor has no twos, the second and third each have one, and beyond that, we add the power of 2 dividing n to the term involving $n!$. This gives the answer to be

$$0 + 1 + 1 + 3 + 3 + 4 + 4 + 7 = \boxed{23}$$

□

Proposed by Matthew Kroesche.

Problem 7. Kelly rolls a fair six-sided die, and Maggie rolls a fair eight-sided die. Compute the probability that the number Kelly rolls is strictly greater than the number Maggie rolls. Express your answer as a common fraction.

Solution. There are $6 \times 8 = 48$ total possible outcomes, and the number of outcomes where Kelly rolls higher than Maggie is equal to the number that Kelly rolls, minus one. So in total, there are $0+1+2+3+4+5 = 15$ such outcomes, so the answer is $\frac{15}{48} = \boxed{\frac{5}{16}}$ □

Proposed by Matthew Kroesche.

Problem 8. Compute 142857×7.

Solution. We see, either by carrying out the multiplication or by recognizing the decimal expansion $\frac{1}{7} = 0.142857142857\ldots = \frac{142857}{999999}$, that the answer is $\boxed{999999}$. □

Proposed by Matthew Kroesche.

Problem 9. The sleepy sloth wakes up on Monday at 6 AM. Every following day, the sloth wakes up at some random time within an hour of the previous day. On Friday, let the earliest hour the sloth could wake up be m AM and the latest be M AM. Compute $M + m$.

Solution 1. Observe that on each day, the earliest the sloth can wake decreases by an hour, and the latest the sloth can wake increases by an hour. Thus, the sum of the least and greatest hours does not change on any particular day, so the answer is $6+6 = \boxed{12}$. □

Solution 2. The latest the sloth could wake up every day increases by an hour, and the earliest the sloth could wake up decreases by an hour. Thus in four days' time, the earliest the sloth could wake up is $6-4 = 2$ AM, and the latest is $6+4 = 10$ AM. The answer is $2+10 = \boxed{12}$. □

Proposed by Nir Elber.

Problem 10. Dylan shuffles a standard 52-card deck and begins drawing cards from it without replacement. The first two cards he draws are both Jacks. What is the probability that the third card he draws is also a Jack? Express your answer as a common fraction.

Solution. The deck now contains 50 cards, 2 of which are Jacks. So the answer is $\frac{2}{50} = \boxed{\frac{1}{25}}$. □

Proposed by Matthew Kroesche.

Problem 11. Eve chooses a positive integer less than 100 which is divisible by 3. Compute the probability that it also even. Express your answer as a common fraction.

Solution. The number of possibilities which Eve could have selected in total is simply the number of integers divisible by 3. This set is

$$3 \cdot 1, 3 \cdot 2, 3 \cdot 3, \ldots, 3 \cdot 33 = 99,$$

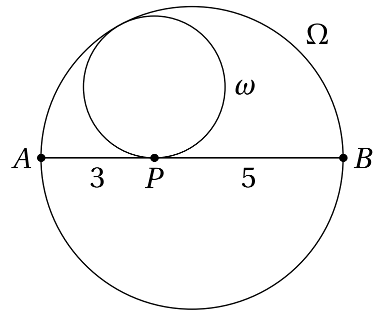
so there are 33 options. However, for Eve's choice to be even, it is necessary and sufficient for Eve's number to be divisible by $2 \cdot 3 = 6$. This set is

$$6 \cdot 1, 6 \cdot 2, 6 \cdot 3, \ldots, 6 \cdot 16 = 96,$$

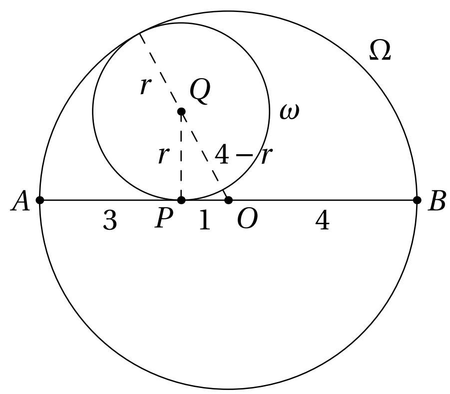
so there are 16 options here. The final probability, then is $\boxed{\frac{16}{33}}$. □

Proposed by Nir Elber.

Problem 12. Circle Ω has diameter AB. Circle ω is tangent to segment AB at P, and internally tangent to circle Ω. If $AP = 3$ and $BP = 5$, compute the radius of circle ω. Express your answer as a common fraction.



Solution. We see that $AB = 8$, so the radius of circle Ω is 4. Let r be the radius of ω. Letting O be the center of Ω and Q be the center of ω, consider triangle OPQ, which has a right angle at P. Then $OP = 4-3 = 1$, $OQ = 4-r$, and $PQ = r$. Thus, by the Pythagorean Theorem, $r^2 + 1 = (4-r)^2 = r^2 - 8r + 16$. Thus $8r = 15$ and $r = \boxed{\frac{15}{8}}$.



□

Proposed by Matthew Kroesche.

Problem 13. Ronda Rousey replaces rosy rocks with Reckless Rick's red rings. Reckless Rick reports 10 red rings, but Ronda Rousey can only replace Reckless Rick's red rings with her rosy rocks at a rate of 3 or fewer red rings for 1 rosy rock. Once all red rings are removed, compute the minimum possible number of rosy rocks at the end.

Solution. This is just $\lceil \frac{10}{3} \rceil = \boxed{4}$ rosy rocks. □

Proposed by Nir Elber.

Problem 14. If a and b are positive real numbers satisfying

$$a - b = 3$$

$$a^2 - b = 33$$

then compute $a^3 - b$.

Solution. Subtracting the two equations, $a^2 - a = 30 \implies (a-6)(a+5) = 0$ so $a = 6$. Then $b = 3$, so $a^3 - b = \boxed{213}$. □

Proposed by Matthew Kroesche.

Problem 15. Alicia rolls a fair twelve-sided die, and Sophie rolls a fair twenty-sided die. Compute the probability that the numbers showing on both dice are prime. Express your answer as a common fraction.

Solution. There are five primes less than 12 $(2,3,5,7,11)$ and three more between 12 and 20 $(13,17,19)$. Thus the probability that the first number is prime is $\frac{5}{12}$ and the probability that the second number is prime is $\frac{8}{20}=\frac{2}{5}$. Thus the probability that both are prime is $\frac{5}{12}\cdot\frac{2}{5}=\frac{2}{12}=\boxed{\dfrac{1}{6}}$. □

Proposed by Matthew Kroesche.

Problem 16. What is the sum of all the three-digit positive integers with the property that each of their digits is either 1, 3, or 6?

Solution. We solve this problem using linearity of expectation. The expected value of a digit is $\frac{1+3+6}{3}=\frac{10}{3}$, so the average value of one of these integers is $\frac{10}{3}(10^2+10+1)=\frac{1110}{3}$. Since there are $3^3=27$ such numbers, their sum is $9\cdot 1110=\boxed{9990}$. □

Proposed by Matthew Kroesche.

Problem 17. If x, y are real numbers such that $(x+1)(y+1)=20$ and $(x+2)(y+2)=40$, compute $(x+3)(y+3)$.

Solution. We have $xy+x+y+1=20$ and $xy+2x+2y+4=40$. Thus, subtracting, $x+y+3=20$ and so $x+y=17$. Then we have

$$(x+3)(y+3)=xy+3x+3y+9=(xy+2x+2y+4)+(x+y)+5=40+17+5=\boxed{62}$$

□

Proposed by Matthew Kroesche.

Problem 18. How many positive integers less than 30 can be written as the product of two distinct primes?

Solution. We see that $2\cdot 3=6$, $2\cdot 5=10$, $2\cdot 7=14$, $2\cdot 11=22$, and $2\cdot 13=26$. ($2\cdot 17=34$ is too big.) Furthermore, $3\cdot 5=15$, $3\cdot 7=21$, and the next two ($3\cdot 11=33$ and $5\cdot 7=35$) are both too big. So there are a total of $\boxed{7}$. □

Proposed by Matthew Kroesche.

Problem 19. Tom writes the numbers 1, 2, 2^2, 2^3, 2^4 and 2^5 on a blackboard. He then selects two and looks at their difference. Compute the number of possible positive differences.

Solution. This is not difficult to bash, but in fact we claim that for each unordered pair $\{a, b\}$ from the set, the positive difference between these two numbers is unique. Indeed, suppose that we have $a\geq b$ and $c\geq d$ such that

$$2^a-2^b=2^c-2^d.$$

We note that the highest power of 2 dividing the left-hand side is 2^b, and the highest power of 2 dividing the right-hand side is 2^d, so we must have $b=d$. Canceling this out gives

$$2^a=2^c,$$

so we must have $a=c$ as well. Thus, the positive differences are unique.

It follows that there is one unique positive difference for each unordered pair, so there are a total of $\binom{6}{2}=\boxed{15}$ total positive differences. Note that choosing the same number twice would not give a positive difference. □

Remark: I want to reward students for trying to think through the logic of why the differences are always unique, but I still want to make the problem bash-able. As a sidenote, the result stated becomes "obvious" if we work in binary, but we can't expect students to see that.

Proposed by Nir Elber.

Problem 20. Matthew has one coin, and Nir has no coins. Rich Farmer Joe will continually give coins randomly to either Matthew or Nir until the total number of coins Matthew and Nir have is divisible by 5. Compute the expected number of coins Matthew gains.

Solution. At each time step, notice that Farmer Joe either gives a coin to Matthew or Nir, and either case, the total number of coins between them increases by 1, always. Thus, the total number of coins follows the sequence 1, 2, 3, 4, 5 and then terminates, so Rich Farmer Joe only gives 4 coins.

Each coin is equally likely to go to either Nir or Matthew, so the expected number of coins Matthew gains is $\frac{4}{2} = \boxed{2}$. □

Remark: Asking for the difference instead of the sum (total) creates for a nice states question and could be an interesting variant on the common expected time questions. In particular, we do not increment at each time step.

Proposed by Nir Elber.

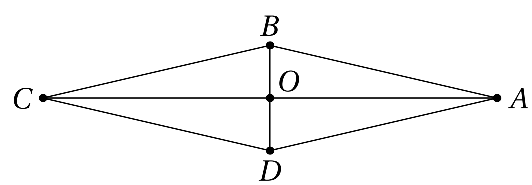
Problem 21. Matthew's favorite primes are $3, 7, 13$, and 17. He randomly chooses two different primes out of these four and sums their cubes. Find the probability that this sum is divisible by 8.

Solution. We know that none of these numbers are even so $p^3 + q^3 \cong p + q \pmod 8$. Since these numbers are also coincidentally $1, 3, 5, 7 \pmod 8$, only 3 and 5 or 1 and 7 work. Thus the answer is $\frac{2}{\binom{4}{2}} = \boxed{\frac{1}{3}}$. □

Proposed by Ethan Liu.

Problem 22. A rhombus has perimeter 20 and area 11. Compute the sum of its diagonals.

Solution. Let the diagonals be d_1 and d_2. Consider the following diagram of the rhombus in the question.



We simply let $AC = d_1$ and $BD = d_2$. As this is a rhombus, the diagonals are perpendicular and bisect each other, so we compute

$$AB = \sqrt{OB^2 + OA^2} = \sqrt{\left(\frac{d_1}{2}\right)^2 + \left(\frac{d_2}{2}\right)^2} = \frac{1}{2}\sqrt{d_1^2 + d_2^2}.$$

It follows that the perimeter is $20 = 4AB = 2\sqrt{d_1^2 + d_2^2}$. We can also compute the area as

$$11 = [ABCD] = [ABC] + [ADC] = \frac{1}{2}d_1 \cdot \frac{1}{2}d_2 + \frac{1}{2}d_1 \cdot \frac{1}{2}d_2 = \frac{1}{2}d_1 d_2.$$

Therefore we have the following system:

$$\begin{cases} 2\sqrt{d_1^2 + d_2^2} = 20 \\ \frac{1}{2}d_1 d_2 = 11 \end{cases} \implies \begin{cases} d_1^2 + d_2^2 = 100 \\ d_1 d_2 = 22 \end{cases}.$$

Keeping in mind that we want to compute $d_1 + d_2$, we note that

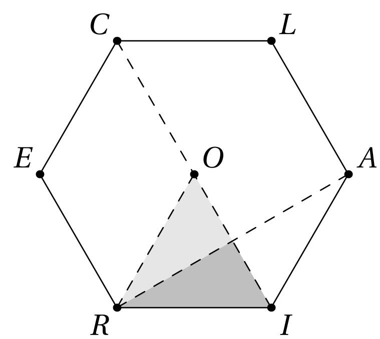
$$(d_1 + d_2)^2 = d_1^2 + d_2^2 + 2d_1 d_2 = 100 + 2 \cdot 22 = 144.$$

Thus, our final answer is $\sqrt{144} = \boxed{12}$. □

Proposed by Nir Elber.

Problem 23. Regular hexagon $CLAIRE$ has area 1. Compute the area of the intersection of triangle AIR and quadrilateral $ERIC$. Express your answer as a common fraction.

Solution. Refer to the diagram below: the equilateral triangle ORI has area $\frac{1}{6}$, since a regular hexagon can be decomposed into six such congruent equilateral triangles. Since $\angle AIR = 120°$ and triangle AIR is isosceles, $\angle ARI = 30°$, so AR bisects $\angle ORI$, and thus also (since ORI is equiateral) bisects segment OI. Then the dark shaded area (which is the intersection of the two polygons) equals the light shaded area, and each is half the area of ORI. Thus the answer is $\frac{1}{2} \cdot \frac{1}{6} = \boxed{\frac{1}{12}}$.



□

Proposed by Matthew Kroesche.

Problem 24. Josh is buying coke and mentos at a store. He pays the cashier \$15, including both the cost and some amount of change. However, the cashier is very tired, and inadvertently swaps the dollar and cent amounts for the purchase. He then proceeds to give the Josh back half the change he needs. What was the original price of the coke and mentos?

Solution. Let the original price of the coke and mentos be $100a + b$. Then the price the cashier charged was simply $100b + a$, and the equation describing the change from the purchase is $\frac{1}{2}(1500 - (100a + b)) = (1500 - (100b + a)$ or $1500 = 199a - 98b$. Let $a = 2b$. Then $1500 = 199a - 98b = 398b - 98b = 300b$ or $b = 5$. Therefore, the original price of the coke and mentos was $\boxed{\$5.10}$. □

Proposed by Joshua Pate.

Problem 25. A point on the graph of $y = x^2$, other than the origin, is twice as far from the x-axis as it is from the y-axis. How far away is it from the origin? Express your answer in simplest radical form.

Solution. The absolute value of the point's y-coordinate (that is, its distance from the x-axis) must be twice the absolute value of its x-coordinate (that is, its distance from the y-axis). Be careful not to switch these two! Thus it lies on the graph of $|y| = 2|x|$, so we have $x^2 = 2|x|$. Since $x \neq 0$, $|x| = 2$ and $y = 4$. Thus the distance to the origin is $\sqrt{2^2 + 4^2} = \sqrt{20} = \boxed{2\sqrt{5}}$. □

Proposed by Matthew Kroesche.

Problem 26. Compute the largest positive integer that is equal to four times the sum of its digits.

Solution. No one-digit integer satisfies this. For two-digit integers, if the integer is $\underline{A}\,\underline{B} = 10A+B$, we have $10A+B = 4A+4B$ and thus $3B = 6A$, so $B = 2A$. This gives $12, 24, 36, 48$ as the possible choices. For three-digit integers, four times the sum of the digits is at most $4 \times 3 \times 9 = 108$, but no positive integer from 100 to 108 has a digit sum higher than $1+0+8 = 9$, so there are no solutions there. For integers having $n \geq 4$ digits, the integer is at least 10^{n-1} but four times the digit sum is at most $36n$, so we can see that the former will be far bigger than the latter. Thus the largest such integer is $\boxed{48}$. □

Proposed by Ethan Liu.

Problem 27. How many distinct positive integers divide $4^{(4^4)}$?

Solution. Write

$$4^{(4^4)} = 4^{256} = (2^2)^{256} = 2^{512}$$

The positive integer divisors of this number are thus $2^0, 2^1, \cdots, 2^{512}$, and so there are $\boxed{513}$ of them. □

Proposed by Matthew Kroesche.

Problem 28. Compute the sum of all distinct prime divisors of $2020^{20+20} \times (20+20)^{2020}$.

Solution. This is just computation. Note

$$2020^{20+20} \cdot (20+20)^{2020} = (20 \cdot 101)^{40} \cdot 40^{2020} = \left(2^2 \cdot 5 \cdot 101\right)^{20} \cdot \left(2^3 \cdot 5\right)^{2020}.$$

Thus, our prime factors are only 2, 5, and 101. The answer is $\boxed{108}$. □

Proposed by Nir Elber.

Problem 29. Wainright writes the number 123456 on a blackboard, and then Eve erases some (possibly zero) digits uniformly at random, so that all possible combinations of digits Eve could have erased are equally likely. Compute the probability that Eve's erasing leaves behind a nonzero number of digits that form an even integer.

Solution. Observe that the total number of possible remaining numbers is 2^6 because each digit 1–6 has 2 options: It is either erased by Eve or not erased by Eve. It remains to count the number of possibilities that are even.

Let d be the last digit not erased. Observe that the condition we want is really that d is even. Thus we have the following cases.

- $d = 6$ occurs with probability $\frac{1}{2}$ because this is just the probability 6 is not erased.
- $d = 4$ occurs with probability $\left(\frac{1}{2}\right)^3$ because this is just the probability 4 is not erased but 5 and 6 are.
- $d = 2$ occurs with probability $\left(\frac{1}{2}\right)^5$ because this is just the probability 2 is not erased but 3–6 are.

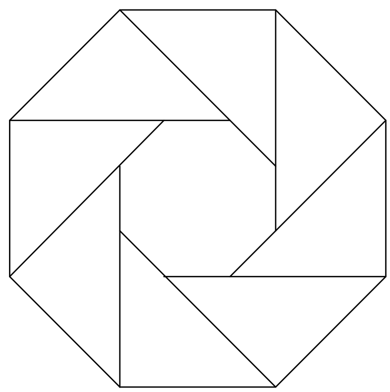
Thus, the final probability is

$$\frac{1}{2} + \frac{1}{8} + \frac{1}{32} = \frac{16+4+1}{32} = \boxed{\frac{21}{32}}.$$

So we are done. □

Proposed by Nir Elber.

Problem 30. The interior of a large regular octagon is partitioned as shown into eight isosceles right triangles and one small regular octagon. Compute the ratio of the area of the large regular octagon to the area of the small regular octagon. Express your answer in simplest radical form.



Solution. Suppose the large octagon has side length 1. Then each right triangle has hypotenuse $\sqrt{2}$, so the small octagon has side length $\sqrt{2}-1$. Thus the ratio of the side lengths is $\frac{1}{\sqrt{2}-1} = \sqrt{2}+1$, and the ratio of the areas is the square of this, which is

$$(\sqrt{2}+1)^2 = 2+2\sqrt{2}+1 = \boxed{3+2\sqrt{2}}$$

□

Proposed by Matthew Kroesche.

Problem 31. A four-digit year $\underline{A}\,\underline{B}\,\underline{C}\,\underline{D}$ is called *powerful* if the product $\underline{A}\,\underline{B} \times \underline{C}\,\underline{D}$ is a perfect square. For example, 2020 is a powerful year since $20 \times 20 = 400$ is a perfect square. What is the next powerful year after 2020?

Solution. If there is another powerful year in the 21st century, we will have $\underline{A}\,\underline{B} = 20 = 2^2 \cdot 5$. In this case, we must have $\underline{C}\,\underline{D}$ equal to 5 times a perfect square. Since $20 = 5 \times 4$, the next larger perfect square after 4 is 9, and $5 \times 9 = 45$. Sure enough, $\boxed{2045}$ is a powerful year since $20 \times 45 = 900 = 30^2$. □

Proposed by Matthew Kroesche.

Problem 32. Find the unique integer between $\frac{2020}{39}$ and $\frac{3180}{61}$.

Solution. There are two options here: one is to do the long division of either fraction, which doesn't take too long. The other is to use the mediant inequality and observe that

$$\frac{2020+3180}{39+61} = \frac{5200}{100} = \boxed{52}$$

is between the two. □

Proposed by Matthew Kroesche.

Problem 33. If

$$x - y = 7$$
$$x^2 - y^2 = 91$$

then compute x.

Solution. Using difference-of-squares factoring, $(x-y)(x+y) = 91$. Since $x - y = 7$, we must have $x + y = \frac{91}{7} = 13$. Then $2x = (x+y)+(x-y) = 13+7 = 20$, so $x = \boxed{10}$. □

Proposed by Matthew Kroesche.

Problem 34. Compute the sum of all positive integers n less than or equal to 27 for which 27 divides $n^2 - 9$.

Solution. The first step is to factor $n^2 - 9 = (n-3)(n+3)$. Note if $27 \mid n^2 - 9$ implies 3 divides n^2, so 3 divides n. Therefore we may write

$$27 \mid (n+3)(n-3) \iff 3 = \frac{27}{9} \;\Big|\; \frac{n+3}{3} \cdot \frac{n-3}{3}$$

because both $n-3$ and $n+3$ are divisible by 3. However, 3 is prime, so 3 divides into the above product if and only if 3 divides into one of the factors. Thus, we have the following two cases.

(i) Note 3 divides $\frac{n+3}{3}$ if and only if 9 divides $n+3$. For n between 1 and 27, this will have solutions for $n+3 \in \{9, 18, 27\}$, namely $n \in \{6, 15, 24\}$.

(ii) Note 3 divides $\frac{n-3}{3}$ if and only if 9 divides $n-3$. For n between 1 and 27, this will have solutions for $n-3 \in \{0, 9, 18\}$, namely $n \in \{3, 12, 21\}$.

It remains to compute the sum of the solutions, which is simply $(6+15+24)+(3+12+21) = (6+21)+(15+12)+(3+24) = 27+27+27 = \boxed{81}$. As a sanity check, one would expect for each n, $27-n$ would also work (note $n^2 - 9 \equiv (-n)^2 - 9 \equiv (27-n)^2 - 9)$, and indeed, this is the case. □

Remark: This problem can be made much harder by adding on only a few more bells and whistles. For example, we could ask for 64 dividing into $n(n+1)(n+2)(n+3)$.

Proposed by Nir Elber.

Problem 35. A shirt costs \$10. The price of the shirt is raised by 30%, and then later it is lowered by 30%. Afterwards, how much does the shirt cost?

Solution. We multiply $10 \cdot \frac{13}{10} \cdot \frac{7}{10} = \frac{91}{10} = 9.1$, so the shirt costs $\boxed{\$9.10}$ now. □

Proposed by Matthew Kroesche.

Problem 36. In the year 2020, there are two months such that the thirteenth day of the month falls on a Friday. One of these months is March. What is the other one?

Solution. We keep adding the remainder when the number of days in each month is divided by 7, waiting until we get something that's a multiple of 7 again. For example, there are 31-28=3 leftover days in March, so April 13th is a Monday. Continuing on in this way, we see that May 13th is a Wednesday, June 13th is a Saturday, July 13th is a Monday, August 13th is a Thursday, September 13th is a Sunday, October 13th is a Tuesday, and $\boxed{\text{November}}$ 13th is a Friday. □

Proposed by Matthew Kroesche.

Problem 37. Harrison has two pizzas, a small 8-inch diameter pizza that is cut into 8 congruent slices, and a large 12-inch diameter pizza that is cut into 10 congruent slices. Compute the ratio of the area of a slice of the large pizza to the area of a slice of the small pizza. Express your answer as a common fraction.

Solution. The large pizza has area equal to $\left(\frac{12}{8}\right)^2 = \frac{9}{4}$ times the area of the small pizza. Thus a single slice of it has area $\frac{9}{40}$ times the area of the small pizza, and thus it has area $\boxed{\frac{9}{5}}$ times the area of a single slice of the small pizza. □

Proposed by Matthew Kroesche.

Problem 38. Compute

$$1024-512+256-128+64-32+16-8+4-2+1$$

Solution. We write this as

$$2^{10}-2^9+\cdots-2^1+2^0=\frac{2^{11}+1}{2+1}=\frac{2049}{3}=\boxed{683}$$

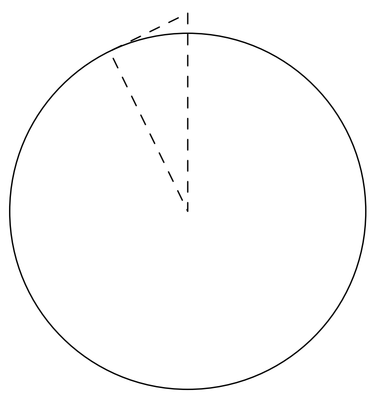
□

Proposed by Matthew Kroesche.

Problem 39. Major Tom is orbiting the Earth at an altitude of 450 miles. Assuming the Earth is a perfect sphere with radius 4000 miles, compute the largest possible distance (in miles) between Major Tom and any point on Earth's surface that he can see from his current location.

Solution. Consider a tangent line from Major Tom to the Earth's surface. Any point farther away from Major Tom than the point where that line meets the Earth's surface is obscured from view by the curvature of the earth. And the distance from Major Tom to the point where the tangent line meets the Earth is found by the Pythagorean Theorem, since tangents are perpendicular to the radii at that point:

$$\sqrt{4450^2-4000^2}=10\sqrt{445^2-400^2}=50\sqrt{89^2-80^2}=50\sqrt{(89+80)(89-80)}=50\sqrt{169\cdot 9}=50\cdot 13\cdot 3=\boxed{1950}$$



□

Proposed by Matthew Kroesche.

Problem 40. Alice tells Bob that the product of two distinct positive integers is 49, but before she tells him their sum, Bob interrupts her and says the sum for her. What is the sum?

Solution. Because $49=7^2$, we can only write $49=1\cdot 49$ or $49=7\cdot 7$; note 7 is prime. Because the integers are distinct, $7\cdot 7$ doesn't work, so we must have that the integers are 1 and 49. Thus, the answer is $1+49=\boxed{50}$. □

Proposed by Nir Elber.

Problem 41. A cube has surface area 42. What is its volume? Express your answer in simplest radical form.

Solution. Letting s be the side length of the cube, we write $6s^2=42$, thus $s^2=7$, $s=\sqrt{7}$, and the volume is $s^3=\boxed{7\sqrt{7}}$. □

Proposed by Matthew Kroesche.

Problem 42. Alice chooses a random set of three distinct nonzero digits, and Eve arranges them into a three-digit integer. Compute the probability that Eve cannot form an even integer. Express your answer as a common fraction.

Solution. Observe that if Alice hands Eve any even digits, then Eve may simply place that digit at the end of the number to guarantee herself an even integer: Any integer which ends with an even digit is even. Thus, the only ways that Eve cannot form an even integer is if Alice happens to hand her all odd digits. There are $\binom{5}{3}$ ways for this to happen, and a total of $\binom{9}{3}$ ways to choose the digits. Thus, our probability is

$$\frac{\binom{5}{3}}{\binom{9}{3}} = \frac{\frac{5\cdot4\cdot3}{3\cdot2\cdot1}}{\frac{9\cdot8\cdot7}{3\cdot2\cdot1}} = \frac{5\cdot4\cdot3}{9\cdot8\cdot7} = \frac{5}{3\cdot2\cdot7}.$$

Thus, our final answer is $\boxed{\frac{5}{42}}$. □

Proposed by Nir Elber.

Problem 43. Sunshine flips a fair coin six times. What is the probability that she sees strictly more heads than tails?

Solution. The probability of getting the same number of heads as tails is $\frac{1}{2^6}\binom{6}{3} = \frac{20}{64} = \frac{5}{16}$. Thus the probability of getting either more heads than tails, or more tails than heads, is $1 - \frac{5}{16} = \frac{11}{16}$. By symmetry, these two probabilities are the same, so the answer is $\boxed{\frac{11}{32}}$. □

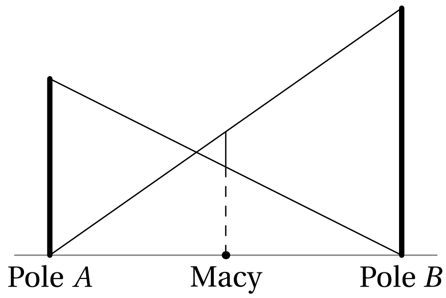
Proposed by Matthew Kroesche.

Problem 44. A squirrel draws 3 circles of radius 1 with their centers at the three vertices of an equilateral triangle with side length 1. A fourth circle is drawn such that the three smaller circles are all inside it and tangent to it. Compute the radius of the fourth circle, expressed as a common fraction in simplest radical form.

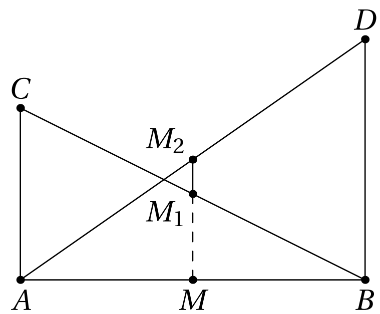
Solution. The centroid of the equilateral triangle is the center of the fourth circle. Thus the radius of the fourth circle is equal to the radius of one of the smaller circles (which is 1) + $\frac{2}{3} * \frac{\sqrt{3}}{2}$, as $\frac{\sqrt{3}}{2}$ is the length of the median of the equilateral triangle. Adding, we get that the radius is equal to $\boxed{\frac{3+\sqrt{3}}{3}}$. □

Proposed by Saskia Solotko.

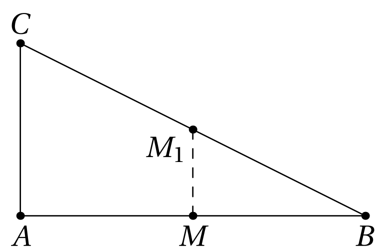
Problem 45. Pole A of height 10 feet and pole B of height 14 feet are stuck in the ground. A cord connects the top of pole A to the bottom of pole B, and another cord connects the top of pole B to the bottom of pole A, as shown. Macy stands at the midpoint between poles A and B and looks straight up. What is the vertical distance between the two cords there?



Solution. For convenience, label the points as follows.



We wish to compute the length M_1M_2, which is also $MM_2 - MM_1$. To compute MM_1, note the following simplified diagram.



Because Macy is looking straight up, $\angle M_1MB = 90° = \angle CAB$, so $\angle M_1BM = \angle CBA$ implies $\triangle CBA \sim \triangle M_1BM$. Thus,

$$\frac{M_1M}{CA} = \frac{MB}{AB} = \frac{1}{2}.$$

It follows that $M_1M = \frac{1}{2}CA = \frac{1}{2} \cdot 10 = 5$ feet.

The exact same argument holds for $\triangle M_2AM \sim \triangle DAB$. From this we once again have

$$\frac{M_2M}{DB} = \frac{AM}{AB} = \frac{1}{2},$$

so $M_2M = \frac{1}{2}DB = \frac{1}{2} \cdot 14 = 7$ feet.

Thus, the final answer is $M_1M_2 = M_2M - M_1M = 7 - 5 = \boxed{2 \text{ feet}}$. □

Remark: Perhaps unsurprisingly, the distance AB is irrelevant to deduce $M_1M_2 = \frac{|BD-AC|}{2}$.

Proposed by Nir Elber.

Problem 46. Given that $3x = 6$, $2x + 2y = 14$, and $x + y + z = 28$, find z.

Solution. The first equation gives us $x = 2$. Plugging in, the next equation gives us $y = 5$, and the final equation gives us $z = \boxed{21}$. □

Proposed by Josiah Kiok.

Problem 47. For real x and y,

$$y^3 = 3x^2 + 3x + 1$$
$$x^3 = 3y^2 + 3y + 1$$

Find $\frac{3x+y}{3y+x}$.

Solution. Reverse the second equation and add it to the first and find that $(x+1)^3 = (y+1)^3$.
Thus, $x + 1 = y + 1$ since x and y are both real.
Thus, the ratio in question is $\boxed{1}$. □

Proposed by Ethan Liu.

Problem 48. Delaney writes the integers "0, 0, 0" on a board, and then her two friends each randomly add 1 to one of Delaney's integers. So, the board could read "0, 2, 0" or "1, 0, 1" at the end. Compute the probability the number 1 is on Delaney's board at the end. Express your answer as a common fraction.

Solution. There are a total of 3^2 ways for Delaney's three friends to choose each of their digits, 3 options for each of her 2 friends; note that the order of addition matters. This counts the total, so it remains to count the number of ways that a 1 appears on the board. We have the following cases.

- Delaney's two friends choose different digits.
  In this case, each of the chosen digits will have exactly one +1 associated with it, so Delaney's board will now contain two 1s. Thus, this case contains a 1.

- Delaney's two friends choose the same digit.
  In this case, the chosen digit will have exactly two +1s associated with it, so the chosen digit will be a 2 while all the other digits are 0. Thus, this case does not contain any 1s.

As is evident, we only care about the case in which Delaney's friends choose different digits. As there are three digits to choose from, there are a total of $3 \cdot 2 = 6$ ways for Delaney's two friends to accomplish this goal (in order).
Thus, our final probability is $\frac{3 \cdot 2}{3^2} = \boxed{\frac{2}{3}}$. □

Proposed by Nir Elber.

Problem 49. Garrett has a bag that contains four red marbles and four blue marbles. He begins to draw marbles from the bag, one at a time, without replacement, and writes down the color of each marble he drew in order. How many different possible sequences of colors could Garrett have written down after he has drawn six of the eight marbles?

Solution. Either 2, 3, or 4 of the six marbles Garrett drew were red. Thus the total number of red-blue sequences is

$$\binom{6}{2} + \binom{6}{3} + \binom{6}{4} = 15 + 20 + 15 = \boxed{50}$$

Alternatively, we can count the total number of ways for Garrett to draw all eight marbles, and then subtract the ways we could have double-counted (which would happen if the last two marbles were two different colors). This would give

$$\binom{8}{4} - \binom{6}{3} = 70 - 20 = \boxed{50}$$

□

Proposed by Matthew Kroesche.

Problem 50. Let $n = 1234$. Compute $\lfloor\sqrt{n+1}\rfloor - \lfloor\sqrt{n}\rfloor$. *($\lfloor x\rfloor$ denotes the greatest integer that is less than or equal to x.)*

Solution. Notice that $n+1 = 1234+1 = 1235$ is divisible by 5 but not 25, so it is not a square. Now let $\lfloor\sqrt{n+1}\rfloor = k$ so that

$$k^2 < n+1 < (k+1)^2.$$

Observe that $k^2 \le n < (k+1)^2$ as well, so we in fact know that $\lfloor\sqrt{n}\rfloor = k$ as well. It follows that $\lfloor\sqrt{n+1}\rfloor - \lfloor\sqrt{n}\rfloor = k - k = \boxed{0}$, so we're done here. □

Proposed by Nir Elber.

Problem 51. Let $a \#\# b = 2$ if $a-b$ is a cube but not a square, 1 if $a-b$ is a square, and 0 otherwise. Find

$$(100\#\#1) + (100\#\#2) + \cdots + (100\#\#100)$$

Solution. The values of $a-b$ being considered are $0, 1, 2, \cdots, 99$. Among these there are two cubes that are not squares ($2^3 = 8$ and $3^3 = 27$) and ten squares (from $0^2 = 0$ to $9^2 = 81$). Thus the sum is $2\cdot 2 + 10\cdot 1 = \boxed{14}$. □

Proposed by Ethan Liu.

Problem 52. If $a \parallel b = \frac{ab}{a+b}$, compute

$$\left(\left(1 \parallel \frac{1}{3}\right) \parallel \frac{1}{8}\right) \parallel \frac{1}{5}$$

Express your answer as a common fraction.

Solution. The fast solution is to see that $\frac{1}{a\parallel b} = \frac{1}{a} + \frac{1}{b}$. This implies the answer is just $\frac{1}{1+3+8+5} = \boxed{\frac{1}{17}}$. However it's also reasonable to just compute the values one at a time; this problem rewards noticing the pattern quickly. □

Remark: The operation in this problem is used in electrical circuits to calculate the equivalent resistance of two resistors in parallel.

Proposed by Matthew Kroesche.

Problem 53. Miss Lilly wants to distribute four identical granola bars among her four students such that no student receives more than two granola bars. In how many ways can she do this?

Solution. There are three cases, based on the number of students that receive two granola bars. If no student receives two granola bars, then each student receives one and there is only **1** way to do this. If one student receives two granola bars, then two other students each receive 1, and there are $4 \times 3 = \mathbf{12}$ ways to choose the student that gets two and the student that gets none. If two students receive two granola bars and the other two receive none, there are $\binom{4}{2} = \mathbf{6}$ ways to choose the two students that receive two. In total, this is $1 + 12 + 6 = \boxed{19}$. □

Proposed by Matthew Kroesche.

Problem 54. Let $A = (-4, 0)$, $B = (0, -5)$, and $C = (4, 5)$. Compute the area of triangle ABC.

Solution. Let $O = (0,0)$ be the origin. We see that O lies inside this triangle, so $[ABC] = [AOB] + [BOC] + [COA]$. We can compute the areas of each of these three triangles using the $\frac{1}{2}bh$ formula, which gives the area of all three triangles to be 10. Thus $[ABC] = 10 + 10 + 10 = \boxed{30}$. ☐

Remark: This can be solved easily enough using the Shoelace formula, but the points are chosen so that hopefully people who haven't seen the Shoelace formula still have a good chance at figuring it out.

Proposed by Matthew Kroesche.

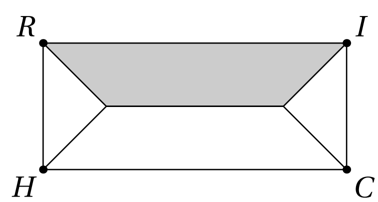
Problem 55. If x, y are positive real numbers such that $x^2 + y^2 = 74$ and $2x^2 + y^2 = 99$, find $x + y$.

Solution. Subtracting the first equation from the second, $x^2 = 99 - 74 = 25$. Since $x > 0$, $x = 5$. Thus $y^2 = 74 - x^2 = 49$, so $y = 7$ since $y > 0$. Thus $x + y = 5 + 7 = \boxed{12}$. ☐

Proposed by Ethan Liu.

Problem 56. Rectangle $RICH$ has $RI = 12$ and $IC = 5$. Compute the area of the set of points in the interior of rectangle $RICH$ that are closer to side $\overline{RI}$ than to any other side of the rectangle. Express your answer as a common fraction.

Solution. The sets of points that are closer to one side of a rectangle than to another adjacent side are separated by the internal angle bisectors (which split each angle into two 45° angles since it's a rectangle). Similarly, the set of points closer to one side than to an opposite side are separated by the line midway between the two sides and parallel to them. Drawing these lines gives the figure below, from which we see that the desired region is a trapezoid with bases of length 12 and 7 and an altitude of length $\frac{5}{2}$. Thus its area is $\frac{5}{2} \cdot \frac{19}{2} = \boxed{\frac{95}{4}}$.



☐

Proposed by Matthew Kroesche.

Problem 57. Find the smallest prime factor of $\frac{15^3-1}{2}$

Solution. It obviously is divisible by 7. Proving it is not divisible by 2 is simple, just note that $15^3 \equiv -1 \pmod 8$ so the number in question is $3 \pmod 4$. Thus the answer is $\boxed{7}$. ☐

Proposed by Ethan Liu.

Problem 58. Compute the number of six-digit integers whose digit sum is divisible by 10.

Solution. Let the integer be $\overline{abc}$ where a, b, and c are the digits. Observe that the condition reads

$$a + b + c \equiv 0 \pmod{10} \iff c \equiv -a - b \pmod{10}.$$

However, $0 \le c < 10$, so given a and b, we can uniquely determine the digit c as $-a - b \pmod{10}$. Thus, it suffices to count the number of pairs (a, b), which is just $9 \cdot 10000 = \boxed{90000}$ because $a \neq 0$. ☐

Proposed by Nir Elber.

Problem 59. The River City High School marching band has 76 students who play the trombone and 110 students who play the cornet. If every student plays at least one of these two instruments, and exactly half of the students in the band play both, compute the total number of students in the River City High School marching band.

Solution. Let n be the number of students who play both instruments. Then $76 - n$ students play the trombone but not the cornet, $110 - n$ students play the cornet but not the trombone, and thus the total number of students in the band is $n + (76 - n) + (110 - n) = 186 - n$. We also know this number equals $2n$, since the n students who play both make up exactly half the band. Thus $3n = 186$, $n = 62$, and the total number of students in the band is $2n = \boxed{124}$. □

Proposed by Matthew Kroesche.

Problem 60. Physicist Phoebe is gluing spherical cows together. Whenever she sees two cows of the same size, she glues them together into one bigger cow, whose size is the sum of the sizes of the two smaller cows. For example, if she has ten size-1 cows, Phoebe would glue them together into five larger size-2 cows, and eventually one size-8 cows with one size-2 cows. What is the minimum number of cows Phoebe must have started with if she leaves four cows of different sizes at the end? (All cows are initially size-1 cows.)

Solution. There are two important points here: All glued spheres contain a perfect power of 2 number of cows (sizes repeatedly double), and whenever there are 2 equally sized spheres, Phoebe will make these into one bigger cow with 2 times the size. In particular, this second point implies that Phoebe will have at most 1 sphere of any one size at the end.

Thus, we can remove the fluff from the problem statement: Because Phoebe ends with 4 spheres (which must be of different sizes) each with a size equal to a power of 2, we may say equivalently that Farmer Frank has

$$2^a + 2^b + 2^c + 2^d$$

cows for some distinct integers a, b, c, d. To minimize this value, we choose the smallest powers of 2 and sum them: $2^0 + 2^1 + 2^2 + 2^3 = 1 + 2 + 4 + 8 = \boxed{15 \text{ cows}}$. □

Remark: Bounding above and asking for maximum might push this into the hard range.

Proposed by Nir Elber.

Problem 61. Kevin and Ramya shared a bag of chips. Kevin ate twice as many chips as Ramya would have eaten if she had eaten twelve fewer chips than Kevin would have eaten if he had eaten half as many chips as Ramya would have eaten if she had eaten four more chips than Kevin would have eaten if he had eaten twice as many chips as he actually did. How many chips did Kevin eat?

Solution. Suppose Kevin ate n chips. Then Ramya would have eaten $2n + 4$ chips if she had eaten four more than Kevin would have eaten, Kevin would have eaten $\frac{1}{2}(2n+4) = n+2$ chips, Ramya would have eaten $(n+2)-12 = n-10$ chips, and Kevin thus ate $2(n-10) = 2n-20$ chips. Thus $2n-20 = n$, so $n = \boxed{20}$. □

Proposed by Matthew Kroesche.

Challenge Round

Calculators and drawing aids are not allowed.

Problem A1. Express
$$\frac{1}{7}+\frac{1}{7\cdot 8}+\frac{1}{7\cdot 8\cdot 13}+\frac{1}{7\cdot 8\cdot 13\cdot 19}+\frac{1}{7\cdot 8\cdot 13\cdot 19\cdot 37}$$
as a common fraction.

Solution. We rewrite as
$$\frac{1}{7}\left(1+\frac{1}{8}\left(1+\frac{1}{13}\left(1+\frac{1}{19}\left(1+\frac{1}{37}\right)\right)\right)\right)$$
and then evaluate the expressions in each parenthesis successively:
$$\frac{1}{19}\left(1+\frac{1}{37}\right)=\frac{1}{19}\cdot\frac{38}{37}=\frac{2}{37}$$
$$\frac{1}{13}\left(1+\frac{2}{37}\right)=\frac{1}{13}\cdot\frac{39}{37}=\frac{3}{37}$$
$$\frac{1}{8}\left(1+\frac{3}{37}\right)=\frac{1}{8}\cdot\frac{40}{37}=\frac{5}{37}$$
$$\frac{1}{7}\left(1+\frac{5}{37}\right)=\frac{1}{7}\cdot\frac{42}{37}=\boxed{\frac{6}{37}}$$

□

Remark: The trick to this problem is inspired by a technique called *Horner's method* from numerical analysis. *Proposed by Matthew Kroesche.*

Problem A2. Suhaas and Connor are working together to paint a fence. Each of them paints at a constant rate. If Suhaas had painted the whole fence himself, it would have taken him x minutes longer than it took them both working together. If Connor had painted the whole fence himself, it would have taken him y minutes longer than it took them both working together. How many minutes does it take Suhaas and Connor to paint the fence together? Express your answer in terms of x and y.

Solution. Let s be the amount of time Suhaas would have taken to paint the fence himself, and let c be the amount of time Connor would have taken to paint the fence himself. We know that the time it actually took them is $\frac{1}{\frac{1}{s}+\frac{1}{c}}=\frac{sc}{s+c}$. Thus,
$$s-\frac{sc}{s+c}=\frac{s^2}{s+c}=x$$
$$c-\frac{sc}{s+c}=\frac{c^2}{s+c}=y$$
Subtracting these two equations gives $s-c=x-y$. Similarly, dividing them and taking the square root gives $\frac{s}{c}=\sqrt{\frac{x}{y}}$. Thus, $s=\sqrt{\frac{x}{y}}c$, and so $\left(\sqrt{\frac{x}{y}}-1\right)c=x-y$, and we see that
$$c=\frac{x-y}{\sqrt{\frac{x}{y}}-1}=\left(\sqrt{x}+\sqrt{y}\right)\sqrt{y}=y+\sqrt{xy}$$
and $s=x+\sqrt{xy}$. Then
$$\frac{sc}{s+c}=\frac{\left(y+\sqrt{xy}\right)\left(x+\sqrt{xy}\right)}{x+2\sqrt{xy}+y}=\sqrt{xy}\cdot\frac{\left(\sqrt{x}+\sqrt{y}\right)\left(\sqrt{x}+\sqrt{y}\right)}{\left(\sqrt{x}+\sqrt{y}\right)^2}=\boxed{\sqrt{xy}\text{ minutes}}.$$

□

Proposed by Matthew Kroesche.

Problem A3. In the election for class president of Preston High School, every student votes for exactly one candidate. Pedro and Summer receive a combined total of exactly 85% of the votes cast, and Pedro receives 43 more votes than Summer. If Pedro wins a strict majority (that is, strictly more than 50%) of the total number of votes, what is the maximum possible number of students who voted for Pedro?

Solution. Suppose Pedro received n votes. Then Summer received $n-43$ votes, so their combined number of votes was $2n-43$. We divide this by 85% (that is, multiply by $\frac{20}{17}$) to get the total number of votes cast. Since this is an integer, we need $2n-43$ to be divisible by 17, or equivalently, n to be 4 less than a multiple of 17. Furthermore, we need n to be strictly greater than half the total number of votes cast. That is,

$$n > \frac{1}{2} \cdot \frac{20}{17}(2n-43) = \frac{10(2n-43)}{17}$$

$$17n > 20n - 430$$

This gives $3n < 430$, so $n < \frac{430}{3} = 143\frac{1}{3}$. The greatest number less than this that is 4 less than a multiple of 17 is $\boxed{132}$. We can check and make sure this works – in this scenario, Summer receives 89 votes, so their combined number of votes is $132+89=221$, and then the total number of votes cast is $\frac{221\cdot 20}{17} = 260$. Sure enough, 132 is just a hair over half of 260, so this does work. □

Remark: This problem rewards going back and checking to make sure your answer actually makes sense.

Proposed by Matthew Kroesche.

Problem A4. Let $\{a_n\}$ be a sequence of the digits after the decimal point of $\frac{1}{995006}$. For example, $a_1 = 0$ because the first digit after the decimal point is 0. Find $a_{15} + a_{21}$.

Solution. $\frac{1}{995006} = \frac{1}{997} - \frac{1}{998}$.
Since $\frac{1}{997} = \sum_{i=0}^{\infty} \frac{1}{1000}\left(\frac{3}{1000}\right)^i$ and $\frac{1}{998} = \sum_{i=0}^{\infty} \frac{1}{1000}\left(\frac{2}{1000}\right)^i$,
Thus the answer is just differences of powers of 3 and powers of 2 with 3 digits of separation. The 15th digit is $81-16=65 \to 5$. The 21st digit is $729-64$ plus the 2 from 2187-128. $667 \to 7$. The answer is $\boxed{12}$. □

Proposed by Ethan Liu.

Problem C1. Define a **stuckset** as a contiguous sequence of characters in a given string, not counting the 0-length sequence. How many stucksets are there in the word BARYBASH with the added condition that no letter appears more than once in each stuckset? *(Two stucksets are the same only if they have the same length and start at the same index within the word.)*

Solution. We see that a stuckset of the word BARYBASH cannot contain either both B's or both A's; otherwise, anything else works. We begin casework on the initial letter. The letter B has 4 possible stucksets. The letter A has 4 possible stucksets. The letters R and downwards have $6,5,4,3,2$, and 1 stuckset respectively. Those sum up to a total of $\boxed{29}$. □

Proposed by Josiah Kiok.

Problem C2. Sly Steve shoots baskets. He shoots up to three baskets at a time before taking a break. If he makes the first basket he shoots, then he just attempts one more. Otherwise, if he misses the first basket he shoots, he calls it a "practice shot" that *doesn't count* and attempts two more baskets. Find Steve's average accuracy over the two baskets that count (that is, the expected number of baskets made to baskets attempted), given the actual probability that he makes a shot is $2/3$. Express your answer as a common fraction.

Solution. Steve hits the first basket with probability 2/3. In this case, there's a 2/3 chance that he makes another basket and his accuracy will be 1, or he'll miss with a 1/3 chance and his accuracy will be 1/2. Otherwise, if he misses the first shot with probability 1/3, The average accuracy is not cheated and his average accuracy will be 2/3. Thus, his actual average accuracy is $\frac{2}{3}(\frac{2}{3}\cdot 1+\frac{1}{3}\cdot\frac{1}{2})+\frac{1}{3}*\frac{2}{3}=\boxed{\frac{7}{9}}$ □

Proposed by Ethan Liu.

Problem C3. Christine randomly chooses a positive integer n from 1 to 100. Lydia randomly chooses a positive integer divisor d of n. Given that $d=33$, compute the expected value of n. Express your answer as a common fraction.

Solution. We know that n is either 33, 66, or 99. In each case, the probability that Lydia chose $d=33$ is $\frac{1}{\tau(n)}$, that is, 1 over the number of divisors of n. Thus, the expected value is

$$\frac{\frac{33}{\tau(33)}+\frac{66}{\tau(66)}+\frac{99}{\tau(99)}}{\frac{1}{\tau(33)}+\frac{1}{\tau(66)}+\frac{1}{\tau(99)}}=\frac{33\left(\frac{1}{4}+\frac{2}{8}+\frac{3}{6}\right)}{\frac{1}{4}+\frac{1}{8}+\frac{1}{6}}=\frac{33}{\frac{13}{24}}=\boxed{\frac{792}{13}}$$

□

Proposed by Matthew Kroesche.

Problem C4. A substring of an integer n is any group of consecutive digits; for example 1, 04, and 104 are all substrings of 104. Furthermore, call a substring *fickle* if no two consecutive digits are both even or both odd; for example 1 and 10 are fickle substrings of 104, but 04 and 104 are not. Compute the expected number of fickle substrings in a randomly chosen 5-digit integer if the leading digit may be 0. Express your answer as a common fraction.

Solution. We do casework on the length of the substring and use linearity of expectation to finish.

Observe that if a substring has length n, then the probability it is fickle is $\frac{1}{2^{n-1}}$. In particular, it doesn't matter what the first digit is (it may be 0), but then the second digit has a $\frac{1}{2}$ probability of being of the correct parity, then the third digit also has $\frac{1}{2}$, and so on for the $n-1$ digits not at the front. Of particular interest is the case of substrings of length 1, which are always fickle.

We can now complete the problem using linearity of expectation. For a length n, there are $6-n$ possible substrings, so the expected number of them is

$$\sum_{k=1}^{5}\frac{6-k}{2^{k-1}}=\frac{5}{1}+\frac{4}{2}+\frac{3}{4}+\frac{2}{8}+\frac{1}{16}=\frac{80+32+12+4+1}{16}=\frac{129}{16},$$

so $\boxed{\frac{129}{16}}$ is our final answer. □

Proposed by Nir Elber.

Problem G1. Let $ABCD$ be a square. Let E,F,G,H lie on sides $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$ respectively such that $AB=2AE=3BF=4CG=5DH$. Find $\frac{[EFGH]}{[ABCD]}$

Solution. Let the side of the square be x. Then, the areas of $[AHE]$, $[BEF]$, $[CFG]$, and $[DGH]$ are $\frac{x^2}{5}$, $\frac{x^2}{12}$, $\frac{x^2}{12}$, and $\frac{3x^2}{40}$. Therefore, the area of $[EFGH]=x^2-\frac{x^2}{5}-\frac{x^2}{12}-\frac{x^2}{12}-\frac{3x^2}{40}=x^2-\frac{53x^2}{120}=\frac{67x^2}{120}$. So, $\frac{[EFGH]}{[ABCD]}=\boxed{\frac{67}{120}}$. □

Proposed by Joshua Pate.

Problem G2. A polyhedron has each face in the shape of either a square, a regular hexagon, or a regular dodecagon, such that a square, a hexagon, and a decagon meet at every vertex of a solid. How many faces does the solid have?

Solution. Let the number of squares in the solid be S, the number of hexagons in the solid be H, the number of decagons in the solid be D, and the number of vertices of the solid be V. Since each vertex corresponds to a vertex of a square, so $S = \frac{V}{4}$. Similarly, $H = \frac{V}{6}$ and $D = \frac{V}{10}$. Also, since each edge of the solid corresponds to two edges of polygons, the number of edges E is $\frac{1}{2}(4S+6H+10D) = \frac{3V}{2}$. Then, by Euler's formula, $V+F-E = V+\frac{31V}{60}-\frac{3V}{2} = \frac{V}{60} = 2$ or $V = 120$. Therefore, the number of faces on the solid is $F = \frac{31V}{60} = \boxed{62}$. □

Proposed by Joshua Pate.

Problem G3. Alex owns two dogs named Bark and Bite. He chains them to opposite sides of a hexagonal doghouse of side length 1 foot. The chains are 2 feet long. Find the area outside of the doghouse which Bark and Bite can both access. Express your answer as a common fraction in terms of π in simplest radical form.

Solution. Note that this is much clearer with a diagram, but essentially once you put them together the two regions are congruent to the intersection of two circles of radius 1 that are 1 unit apart. Thus, the area is $\frac{2\pi}{3} - \frac{\sqrt{3}}{2} = \boxed{\frac{4\pi - 3\sqrt{3}}{6}}$. □

Proposed by Ethan Liu.

Problem G4. In triangle $\triangle ABC$, let the perpendicular bisector of side $\overline{AB}$ intersect line $\overleftrightarrow{AC}$ at a point D and similarly define E to be the intersection of the perpendicular bisector of side $\overline{AC}$ with line $\overleftrightarrow{AB}$. If $AB = 6$, $AE = 8$,and $AC = 9$, find the length of segment $\overline{AD}$. Express your answer as a common fraction.

Solution. Since we are given 3 lengths all sharing an endpoint, it seems that Power of a Point could be useful. We want to prove that B, E, C, D are concyclic, but doing so by itself seems hard; hence, we are then motivated to find a fifth point that also lies on the circle: namely, the circumcenter O. At this point, the details aren't too hard to flesh out. Let M and N be the midpoints of $\overline{AB}$ and $\overline{AC}$, respectively. We can then find that

$$\measuredangle ODC = \measuredangle ADM = 90 - \measuredangle BAC = \measuredangle OBC,$$

where angles are directed, so O, D, C, B are concyclic; an identical angle chase then also shows that O, E, C, B are concyclic. Now, applying Power of a Point finishes the problem, giving us $AD = \frac{AB \cdot AE}{AC} = \boxed{\frac{16}{3}}$. □

Remark: The intersections of the tangents at B and C w.r.t. the circumcircle of $\triangle ABC$ also lie on this circle, and so does the center of the spiral similarity mapping AB to AC!

Proposed by Albert Zhu.

Problem N1. What is the largest positive integer $b \geq 3$ such that 12_b divides 2020_b in base b?

Solution. We write $2020_b = 2b^3 + 2b$ and $12_b = b + 2$. Carrying out the polynomial division gives

$$\frac{2b^3+2b}{b+2} = \frac{(2b^3+4b^2)-4b^2+2b}{b+2} = \frac{(2b^3+4b^2)-(4b^2+8b)+10b}{b+2}$$

and this works out to

$$2b^2 - 4b + 10 - \frac{20}{b+2}$$

This is an integer if and only if $b+2$ divides 20. The largest positive integer b for which this is true is when $b+2 = 20$, or $b = \boxed{18}$. □

Proposed by Matthew Kroesche.

Problem N2. Let $C(N)$ be equal to one more than the smallest prime factor of a positive integer $N > 1$. Collatz picks a positive integer greater than 1 and writes it on a board. Every minute, he replaces the number on the board N with $C(N)$. Compute the largest number of minutes before Collatz writes 4.

Solution. This problem is not as interesting as it looks. We have the following cases.

- If N is three, then Collatz immediately writes down $3 + 1 = 4$ after the first minutes.
- If N is even, then the smallest prime factor is 2. One more is 3, so after 1 minute, Collatz writes 3. Using the previous case, the process ends after 2 minutes.
- IF N is odd but greater than 3, then let its smallest prime factor be p. (Note $N > 1$.) After one minute, Collatz writes down $p + 1$, which is even, so we default to the previous case. The process ends in 3 minutes.

Thus, the largest number of minutes is $\boxed{\text{3 minutes}}$. □

Remark: I don't actually know what happens if we choose one more than the largest prime factor.
Proposed by Nir Elber.

Problem N3. Sylvia thinks of a two-digit positive integer. Scott reverses the order of the two digits to form another two-digit positive integer, which is bigger than Sylvia's integer, and notices that the product of their two integers is 4032. Compute Sylvia's integer.

Solution. We factor $4032 = 2^6 \cdot 3^2 \cdot 7$. Since the reversal of a number is divisible by 3 if and only if the original is (since we're looking at the sum of the digits) each of their two numbers must be divisible by 3 exactly once. Furthermore, exactly one of them is divisible by 7, and so divisible by 21. The only possibilities for this number then are $21, 42, 63, 84$. We rule out 63 since it's divisible by 3 twice. We see that $\frac{4032}{21} = 2^6 \cdot 3 = 192$ which is too big, and dividing by 2, $\frac{4032}{42} = 96$ doesn't work either. So the only possibility left is $\frac{4032}{84} = 48$ which works. Since Sylvia's number is smaller than Scott's, her number must be $\boxed{48}$. □

Proposed by Matthew Kroesche.

Problem N4. Diego is playing basketball. Each basket he shoots is worth either two points or three points. Over the course of the game, he makes more two-point shots than three-point shots, but he does make at least one three-point shot. At the end, Diego observes remarkably that his score is numerically equal to the percentage of his baskets that were two-point shots. Compute Diego's score.

Solution. Let s be Diego's score, and let n be the total number of baskets he shot. Then since $\frac{s}{100}$ is the fraction of his baskets that were worth two points, we have

$$2n\frac{s}{100} + 3n\left(1 - \frac{s}{100}\right) = s$$
$$300n - ns = 100s$$

So $300 - s$ divides $100s$. Since $300 - s$ also divides $100(300 - s) = 30000 - 100s$, it must be true that $300 - s$ divides 30000. Since he made more two-point shots than three-point shots, but still made at least one three-point shot, we have $50 < s < 100$, and s is an integer since his total score must be an integer. So we are looking for a divisor of 30000 that is strictly between 200 and 250. Since $30000 = 200 \cdot 150 = 250 \cdot 120$, we can quickly check numbers in between to find that the only factor in the range is 240, which corresponds to $s = \boxed{60}$. □

Proposed by Matthew Kroesche.

Tiebreaker Round

Problem 1. Four boys (Andrew, Bob, Cadmus, and Doug) are playing a pointing game. Each player points to another player that is not himself. What is the number of ways that the players can point at each other such that there exists no pair of two players that are pointing to each other?

Solution. Denote a cycle of players as a group of players $P_1, P_2 ... P_n$ such that P_1 is pointing to P_2, P_2 is pointing to P_3... and finally, P_n is pointing to P_1. Since 1-cycles and 2-cycles are disallowed, there must only be 3-cycles and higher, as well as players pointing to said cycles. We then begin casework on the largest cycle.

3-Cycle: There are four ways to choose the player not in the cycle. Of the three players in the cycle, there are two ways to choose their orientation. (This can be seen by arranging them in a triangle, leaving clockwise and counterclockwise as the only two options). The remaining player must point to one of the three players. This gives us $4 \cdot 2 \cdot 3 = 24$ possibilities.

4-Cycle: There are three ways to choose the target a given player A points to (denote him player B). There are then two ways to choose the target of player B (denote him player C), and one way to choose the target of player C (denote him player D), and one way to choose the target of player D. Thus, there are $3 \cdot 2 \cdot 1 \cdot 1 = 6$ possibilities if all players are in one cycle.

This gives us $\boxed{30}$ possibilities overall. □

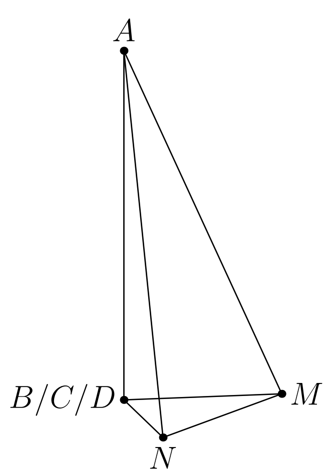
Proposed by Josiah Kiok.

Problem 2. Arie the Ant is at the midpoint of a yardstick. He begins walking to the right, but every time the total number of inches he has walked (in both directions combined) is equal to a positive perfect cube, he turns around. How many total inches has Arie traveled at the instant when he falls off the end of the yardstick?

Solution. Arie starts at the 18 inch mark. He goes one inch to the right, to 19 inches, then turns around and goes $8 - 1 = 7$ inches back to the 12 inch mark. Then he turns around again and goes $27 - 8 = 19$ inches to the right, ending up at the 31 inch mark. Finally, Arie turns around and walks to the left for $64 - 27 = 37$ inches. However, after he has traveled 31 of those 37 inches, he falls off the end at the zero-inch mark. Thus, in total, he has traveled $27 + 31 = \boxed{58}$ inches. □

Proposed by Matthew Kroesche.

Problem 3. Let $ABCD$ be a square of side length 4, and let M and N be the midpoint of BC and CD respectively. Let the square $ABCD$ be folded along the lines AM, AN, and MN so that a tetrahedron is formed. Find the volume of the tetrahedron, expressed as a common fraction.



Solution. Let the point on the top of the tetrahedron be T, and the midpoint of MN be P.Take a cross-section of the tetrahedron through the points, T, P, and A. The triangle formed has sides 4, $\sqrt{2}$, and $3\sqrt{2}$ and area $2\sqrt{2}$. Since this is a right triangle, the height of both the triangle and the tetrahedron is $2[ATP]/AP = \frac{4\sqrt{2}}{3\sqrt{2}} = \frac{4}{3}$. The base of the tetrahedron, $[DMN]$ is $[ABCD]-[ABM]-[ADN]-[CMN]$ or $16-4-4-2=6$. Therefore, the volume of the tetrahedron is $\frac{1}{3}*\frac{4}{3}*6=\boxed{\frac{8}{3}}$. □

Proposed by Joshua Pate.