2021 AMC Practice MATHCOUNTS Solutions

Austin Math Circle

16 January 2021

Contents

Sprint Round

Problem 1. How many positive integers strictly less than 100 are either prime or composite, but not both?

Solution. One is neither prime nor composite, so the integers we want are 2-99, and so the answer is $\boxed{98}$. ■

Proposed by Pierce Lai.
Category: NT

Problem 2. If 640 acres is a square mile, and a mile is 1600 meters, how many square meters is an acre?

Solution. A square mile would be $1600^2 = 2560000$ square meters, so an acre would be $2560000/640 = \boxed{4000}$ square meters. ■

Proposed by Pierce Lai.
Category: Algebra

Problem 3. Anastasia and Bananastasia are driving through a circular roundabout. Anastasia starts at one point of the circle, and travels at a constant speed along the circumference counterclockwise to the point diametrically opposite her starting point. Meanwhile, Bananastasia starts at the same point as Anastasia, but drives straight across the roundabout directly to the other side, also at a constant speed. If they start at the same time and finish at the same time, Anastasia must be traveling x times as fast as Bananastasia. Compute the nearest integer to $10x$.

Solution. Now x is the ratio of the semiperimeter to the diameter of the roundabout, which is $\frac{\pi}{2}$. Thus $10x = 5\pi$. Since $\pi \approx 3.14$, $5\pi \approx 15.7$, so the nearest integer to it is $\boxed{16}$. ■

Proposed by Pierce Lai.
Category: Geometry

Problem 4. Gordon Ramsay and Chef Rush are in a cooking contest which consists of a number of matches. If Rush has a 60% chance of winning each match, then the probability that Ramsay will win the first three matches is p%. Compute $100p$.

Solution. Ramsay has a .4 chance of winning each match, so the chance of winning all three is $4^3 = .064$, which is 6.4%. Thus the answer is $\boxed{640}$. ■

Proposed by Pierce Lai.
Category: Combo

Problem 5. What is the largest number of equilateral triangles of side length 1 that can fit inside a regular hexagon of side length 3 without overlapping?

Solution. 6 unit equilateral triangles can fit in a regular unit hexagon, so scaling that up by 3^2 gives $\boxed{54}$. ■

Proposed by Pierce Lai.
Category: Geo

Problem 6. Working alone, Person A can fill an Olympic-sized swimming pool with concentrated molasses in 7000 years. Person B can do it in 5600 years, while Person C can do it in just 4000 years. How many years will it take for all three of them working together to fill the pool?

Solution. Each year, the pool is filled by $1/7000 + 1/5600 + 1/4000 = (4 + 5 + 7)/28000 = 1/1750$, so the answer is $\boxed{1750}$ years. ■

Proposed by Pierce Lai.
Category: Algebra

Problem 7. Alex owns a farm where he raises chickens and octopi. Each chicken has one brain and one heart, while each octopus has nine brains and three hearts. If Alex has 168 brains and 96 hearts from his chickens and octopi, how many of those hearts are octopi hearts?

Solution. If c is the number of chickens and p is the number of octopi, then we get the system of equations

$$c + 9p = 168$$

$$c + 3p = 96$$

Solving gives $c = 60$ and $p = 12$, so the number of octopi hearts is $3 \times 12 = \boxed{36}$. ■

Proposed by Pierce Lai.
Category: Algebra

Problem 8. Alex is walking through a hallway. Every minute, there is a $1/4$ chance of a wild heff appearing, and a $1/3$ chance of a wild matthew appearing. (It is not possible for both to appear at the same time.) If the probability that Alex's first encounter is with a wild matthew and not a wild heff is expressed in reduced form as $\frac{m}{n}$, what is $100m + n$?

Solution. The total chance per minute of an encounter is $1/3 + 1/4 = 7/12$, so the chance that that encounter is a matthew is $1/3/(7/12) = 4/7$, so our final answer is $\boxed{407}$. ■

Proposed by Pierce Lai.
Category: Combo

Problem 9. Let V be the volume of a regular octahedron with side length 2. Then if $V = \sqrt{\frac{m}{n}}$, where the fraction $\frac{m}{n}$ is in lowest terms, compute $100m + n$.

Solution. We can chop the octahedron into two square pyramids. Each pyramid has a base of area 4 and a height of $\sqrt{2}$ (since one diagonal has length $2\sqrt{2}$, and a height is half of a diagonal). Hence, the volume of the octahedron is $2 \cdot 4 \cdot \frac{\sqrt{2}}{3} = \frac{8\sqrt{2}}{3} = \sqrt{\frac{128}{9}}$. So we answer $\boxed{12809}$. ■

Proposed by Pierce Lai.
Category: Geo

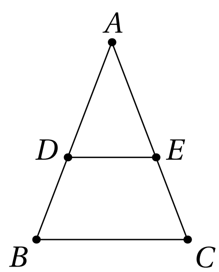
Problem 10. Jay's high school has a large number of students. He knows that 60% of them are boys, the other 40% of them are girls, and that 40% of the boys prefer pancakes over waffles. If overall, 40% of the students prefer waffles over pancakes (and no one is undecided) then what integer percentage of the girls prefer pancakes over waffles?

Solution. We see that the boys contribute $0.6*(1-0.4) = 36\%$ to the overall 40% waffle over pancake statistic, so the girls must contribute 4% total. Thus, the answer is $1-0.04/0.4 = \boxed{90}$ percent. ■

Proposed by Pierce Lai.
Category: Algebra

Problem 11. Isosceles triangle $\triangle ABC$ has $AB = AC = 7$ and $BC = 5$. Points D and E are chosen on sides AB and AC respectively so that $BD = DE = EC$. If the length of segment BD equals the reduced fraction $\frac{m}{n}$, then compute $100m + n$.

Solution. Note the diagram.



Because $AD = AB - BD = AC - EC = AE$, we see that $\triangle ADE$ is A-isosceles. Further, $\angle A$ matches in both, so $\triangle ADC \sim \triangle ABC$ by SAS similarity. Thus,

$$\frac{AD}{AB} = \frac{DE}{BC}.$$

Plugging in, this says $\frac{7-x}{7} = \frac{x}{5}$, which is $35-5x = 7x$ after clearing fractions. Finally, we see $x = \frac{35}{12}$, so we answer $\boxed{3512}$. ■

Proposed by Nir Elber.
Category: Geometry

Problem 12. What is the greatest common divisor of 437 and 589?

Solution. Using two rounds of Euclid's algorithm gives $589 - 437 = 152$, $437 - 3*152 = -19$. We notice that $152 = 19*8$, so the answer is $\boxed{19}$. $(437 = 19*23 = 21^2 - 2^2; 589 = 19*31)$ ■

Proposed by Pierce Lai.
Category: NT

Problem 13. A contest has 8 questions. Each question is written by Nir with probability $\frac{1}{3}$, and each of Nir's questions is about number theory with probability $\frac{1}{2}$. The remaining questions are written by Matthew, each of which is about number theory with probability $\frac{1}{4}$. (All probabilities are assumed to be independent.) The expected number of number theory questions on the contest is a reduced fraction $\frac{m}{n}$. Compute $100m + n$.

Solution. The probability any given question is about number theory is

$$\underbrace{\frac{1}{3}\cdot\frac{1}{2}}_{\text{Nir's}}+\underbrace{\frac{2}{3}\cdot\frac{1}{4}}_{\text{not Nir's}}=\frac{1}{6}+\frac{1}{6}=\frac{1}{3}.$$

Summing with linearity of expectation, the expected number of number-theory questions is $\frac{8}{3}$, so we answer $\boxed{803}$. ■

Proposed by Nir Elber.
Category: Combo

Problem 14. Theodore writes down all positive divisors of 10! divisible by 3. Eve then erases all of the even divisors. Compute the number of remaining divisors.

Solution. We can factor this as $2^{1+2+1+3+1}\cdot 3^{1+1+2}\cdot 5^{1+1}\cdot 7 = 2^8\cdot 3^4\cdot 5^2\cdot 7$. All factors have the form $2^a\cdot 3^b\cdot 5^c\cdot 7^d$. We need $a=0$ to ensure that Eve didn't erase it, but $b>0$ to ensure that Theodore wrote it down. Otherwise, we are left with $1\cdot 4\cdot 3\cdot 2=\boxed{24}$ divisors remaining. ■

Proposed by Nir Elber.
Category: NT

Problem 15. Madeline is climbing a 2000-meter tall mountain. She starts climbing from the base of the mountain at noon, and ascends at a constant speed of 10 meters per minute. After she has been climbing for some time, the wind begins blowing and slows her ascent to 6 meters per minute. If it is 4:00 PM on the same day when Madeline reaches the summit, how many minutes past noon was it when the wind started blowing?

Solution. Let t denote the number of minutes Madeline climbed before the wind started blowing. Then she was climbing for $240-t$ minutes after the wind started blowing. The total distance she climbed was 2000 meters, so we have

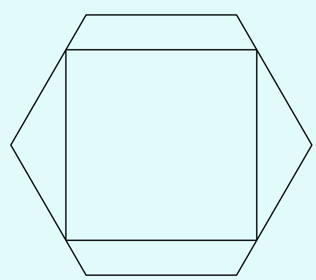
$$10t+6(240-t)=2000$$

$$10t+1440-6t=2000$$

Thus $4t=560$, and so $t=140$. So the wind started blowing $\boxed{140}$ minutes past noon. ■

Proposed by Matthew Kroesche.
Category: Algebra

Problem 16. In the diagram below, a square is inscribed in a regular hexagon of side length 1, and two of its opposite sides are parallel to two of the sides of the hexagon. The area of the square is $m-\sqrt{n}$, where m, n are positive integers. Compute $100m+n$.



Solution. Let the square have side length s. Then consider the line segment joining the leftmost vertex of the hexagon to the rightmost. Its length is twice the side length of the hexagon, which is 2. But we can also write its length as $s+\frac{s}{\sqrt{3}}$, where the s is the length of the part inside the square, and the two other segments each have length $\frac{s}{2\sqrt{3}}$ due to 30-60-90 right triangles. Thus

$$s+\frac{s}{\sqrt{3}}=2$$

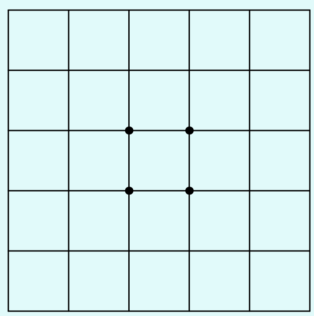
so

$$s=\frac{2}{\frac{1}{\sqrt{3}}+1}=\frac{2\sqrt{3}}{\sqrt{3}+1}=\sqrt{3}(\sqrt{3}-1)=3-\sqrt{3}$$

and the area of the square is $(3-\sqrt{3})^2=12-\sqrt{108}$. So we answer $\boxed{1308}$. ■

Proposed by Pierce Lai.
Category: Geometry

Problem 17. How many paths, following the segments of the grid below and moving only up or right on each step, pass through one of the four vertices of the center square?

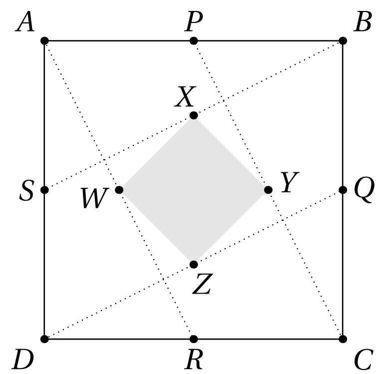


Solution. A path that passes through either the bottom left or top right of the four vertices must also pass through one of the other two, and in fact it must pass through exactly one of the other two. Thus the question is just how many paths pass through the bottom right vertex of the center square, plus how many paths pass through the top left vertex. By symmetry, these two numbers are the same. Using standard block-walking techniques, the answer is thus $2\binom{5}{2}\binom{5}{3}=\boxed{200}$. ■

Proposed by Joshua Pate.
Category: Combo

Problem 18. Let $ABCD$ be a unit square. Further, let P, Q, R, S be the midpoints of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$ respectively. Finally, let W, X, Y, Z be the midpoints of $\overline{AR}$, $\overline{BS}$, $\overline{CP}$, $\overline{DQ}$ respectively. If the area of quadrilateral $WXYZ$ is $\frac{m}{n}$ when expressed as a reduced fraction, find $100m+n$.

Solution. Note the diagram.



Notice that rotating the figure by 90° sends A to B to C to D to A, so because we are only taking midpoints here, the entire figure will be laid on top of itself. It follows that $WXYZ$ has 90° rotational symmetry, and because it is a quadrilateral, it follows that $WXYZ$ must be a square.

It will be enough to compute the length of a diagonal of the square, say XZ. Note that $\overline{ZR}$ by midsegments is parallel to $\overline{QC}$, which is perpendicular to $\overline{CD}$ by square. So $\overline{ZR}$ is the perpendicular bisector of $\overline{CD}$. Rotating the argument 180°, we see that $\overline{PX}$ is the perpendicular bisector of $\overline{AB}$. However, $\overline{PR}$ is the perpendicular bisector of both of these, so P, Q, Z, R all lie on the same line.

So we see that $XZ = PR - PX - ZR$. By midsegments, we see

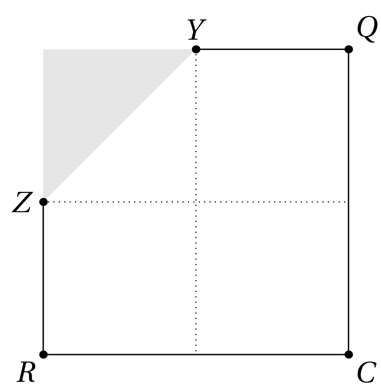
$$PX = \frac{1}{2}AS = \frac{1}{4}AD = \frac{1}{4},$$

and similar holds for ZR. Noting $PR = 1$, we get that $XZ = 1 - \frac{1}{4} - \frac{1}{4} = \frac{1}{2}$. So the area of the square is

$$[WXYZ] = \frac{1}{2}d^2 = \frac{1}{2}XZ^2 = \frac{1}{8},$$

which is what we wanted. So we answer $\boxed{108}$. ■

Solution. The presented solution is lengthy in order to justify visual observations which appear out of the diagram. In practice, it is likely contestants convince themselves of the answer using something like the dissection argument below. By symmetry, it suffices to find the proportion of $WXYZ$ covering the lower-right quadrant:



Here, $WXYZ$ covers an isosceles right triangle, which is half of a quadrant of the diagram. This multiplies out to $\frac{1}{8}$ of the total area, which is what we wanted. ■

Proposed by Nir Elber.
Category: Geometry

Problem 19. Alex starts with a bar of chocolate. Every second, he flips a fair coin, and if it comes up heads, he divides every chocolate bar in front of him into three smaller chocolate bars. (If it comes up tails, he does nothing.) Compute the expected number of chocolate bars he has after five seconds.

Solution. Every step, on average, Alex doubles the number of chocolate bars. Since this is a linear function that applies for any step, we can just keep applying it to get the expected value, which is $\boxed{32}$. ■

Proposed by Ethan Liu.
Category: Combo
Remark: I think this problem is quite easy but conceptually difficult (Also, I could be wrong, but I doubt it). I have a problem that's quite like it except that it's conceptually easier, but I'll put both down at some point.

Problem 20. Call a positive integer *excellent* if the largest positive prime power that divides it is 9. For example, 18 is excellent because its divisors are 1, 2, 3, 6, 9, 18; and out of these, 9 is the largest one that is a prime power. Compute the sum of all excellent positive integers.

Solution. Let n satisfy the condition of the question. Observe that n cannot have any prime divisor larger than 9, so we limit our view to numbers of the form

$$n = 2^{a_1} \cdot 3^{a_2} \cdot 5^{a_3} \cdot 7^{a_4}.$$

We have to have $a_2 = 2$ so that 9 actually divides into n. Because no prime power may exceed 9 in size, $a_1 \le 3$, $a_3 \le 1$, and $a_4 \le 1$. Notice that all of these numbers appear in the product

$$\left(1 + 2^1 + 2^2 + 2^3\right) \cdot 9 \cdot (1+5) \cdot (1+7)$$

by simply expanding out the factorizations. This evaluates to $15 \cdot 9 \cdot 6 \cdot 8 = 30 \cdot 9 \cdot 3 \cdot 8 = 90 \cdot 72 = \boxed{6480}$. ■

Proposed by Nir Elber.
Category: NT

Problem 21. In how many ways can six identical $3 \times 3 \times 1$ blocks be packed into a fixed $3 \times 3 \times 6$ box? (Count two ways as different if there is some block occupying some $3 \times 3 \times 1$ region of space in one arrangement, but no block occupying that exact same region of space in the other.)

Solution. Clearly all the blocks will have to be aligned with the sides of the box, or else there will be wasted space. We do casework based on how many of the blocks have their one-unit side parallel to the box's six-unit side. If all of them do, there's only one way to put them in, all oriented the same way next to one another. Suppose a block doesn't; that is, its one-unit side is oriented parallel to one of the three-unit sides of the box. Then we must make use of the space next to that block; the only way to do that is to stack it together with two other blocks oriented the same way to form a $3 \times 3 \times 3$ cube. Thus, either zero, three, or six blocks are oriented "sideways" like this.

If three of them are sideways, then there are four ways to decide where to place the $3 \times 3 \times 3$ cube among the other three stacked blocks. Then there are two ways to decide which axis the cube is oriented along (there are three axes, but if the blocks in the cube have their short side parallel to the box's long side, it's just the same as the first arrangement we've already counted). So in total there are $4 \times 2 = 8$ arrangements that have three blocks sideways and three not.

Finally, suppose all six are sideways, so that we have two cubes next to each other. There are two ways to orient each one, for a total of $2 \times 2 = 4$ arrangements. So the answer to the question is $1 + 8 + 4 = \boxed{13}$. ■

Proposed by Matthew Kroesche.
Category: Combo, Geometry

Problem 22. What is the product of all positive integers $a < 10$ where $a! + 1$ is a perfect square?

Solution. Small values of a can be checked manually and quickly.

a	$a!+1$
1	2
2	3
3	7
4	$25 = 5^2$
5	$121 = 11^2$
6	721

So far we have $a = 4$ and $a = 5$ work. The remaining a can be checked manually as well, but there are some tricks we can use. We observe that

$$10! = (1 \cdot 10) \cdot (2 \cdot 6) \cdot (3 \cdot 4) \cdot (5 \cdot 9) \cdot (7 \cdot 8) \equiv 10 \pmod{11}.$$

From this, we can say that $9! + 1 \equiv 2 \pmod{11}$ and $8! + 1 \equiv 6 \pmod{11}$. However, the squares modulo 11 are $\{0^2, 1^2, 2^2, 3^2, 4^2, 5^2\} \equiv \{0, 1, 4, 9, 5, 3\}$, so $a = 8$ and $a = 9$ fail.

It remains to check $a = 7$. It is possible to see directly that $7! + 1 = 5041 = 71^2$, which works. As some motivation, we remark that we would be solving

$$7! = k^2 - 1 = (k-1)(k+1).$$

It follows $k \equiv \pm 1 \pmod 7$. Having computed $6! > 700$, we see $\sqrt{7!+1} > 70$ but should be reasonbly close, which leaves onlt $k = 71$ to check. Regardless, we see $2 \cdot 5 \cdot 7 = \boxed{140}$ is our final answer. ■

Proposed by Joshua Pate.
Category: NT

Problem 23. Gina chooses four random positive one-digit integers a, b, c, and d, and she finds that $a \neq c$. The probability that the slope of the line connecting (a, b) and (c, d) is positive is written as a reduced fraction $\frac{m}{n}$. Compute $100m + n$.

Solution. Note that if the slope of the line connecting (a, b) and (c, d) is positive, then exchanging the y coordinates tells us that the slope of the line connecting (a, d) and (c, b) is negative. (This is the key observation.) It follows that the probability of the slope being positive and the probability of the slope being negative are equal to each other. Let this probability be p.

We see that $1 - 2p$ is the probability that the slope is neither negative nor positive, i.e. zero. However, given that $a \neq c$, the slope is 0 if and only if $b = d$, which will occur with probability $\frac{1}{9}$—no matter what b is, d has a one in nine chance of being equal. Thus,

$$1 - 2p = \frac{1}{9},$$

so $p = \frac{4}{9}$ after rearranging, and our answer is $\boxed{409}$. ■

Proposed by Nir Elber.
Category: Combo

Problem 24. Let b be a random integer from -7 to 3, and let a be a random integer from 1 to 3, inclusive. If the probability that the smaller solution for x in the quadratic equation $ax^2 + bx = a + b$ is 1 is the reduced fraction $\frac{m}{n}$, compute $100m + n$.

Solution. We can factor the quadratic as $ax^2 + bx - (a+b) = (x-1)(ax + (a+b))$. Hence, we need $-\frac{a+b}{a} \geq 1$, or $\frac{a+b}{a} \leq -1$, which is equivalent to $\frac{b}{a} \leq -2$. For $a = 1$, the integers that work go from -7 to -2, inclusive. For $a = 2$, that range is -7 to -4. For $a = 3$, the range is -7 to -6. In total, $6+4+2 = 12$ of the 33 ordered pairs work. Hence, the probability is $\frac{12}{33} = \frac{4}{11}$, which makes our final answer $\boxed{411}$. ∎

Proposed by Ethan Liu.
Category: Algebra, Combo
Remark: vieta jumping in mathcounts lol

Problem 25. If a right triangle has perimeter 20 and area 16, the length of its hypotenuse is the reduced fraction $\frac{m}{n}$. Compute $100m + n$.

Solution. Let the legs of the triangle be a and b, and let the hypotenuse be c. Then

$$a + b + c = 20$$

$$ab = 32$$

$$a^2 + b^2 = c^2$$

Now

$$a^2 + b^2 = (a+b)^2 - 2ab = (20-c)^2 - 64 = c^2 - 40c + 336$$

Thus

$$c^2 - 40c + 336 = c^2$$

and so $40c = 336$. Thus

$$c = \frac{336}{40} = \frac{42}{5}$$

and we did not need to find the values of a and b. Our answer is $\boxed{4205}$. ∎

Proposed by Matthew Kroesche.
Category: Algebra, Geometry

Problem 26. Yolanda is yelling positive integers. The first two numbers she yells are 1 and 3, and every following integer Yolanda yells is the product of the previous two. What is remainder when the 100th yelled number is divided by 32?

Solution. The sequence of positive integers she's yelling are $y_0 = 1$, $y_1 = 3$ and then $y_{n+2} = y_{n+1}y_n$ so that we want $y_{99} \pmod{10}$. Computing some small values, we see that the sequence goes

$$3^0, 3^1, 3^1, 3^2, 3^3, 3^5, \dots.$$

In particular, we can show that $y_n = 3^{F_n}$ where F_n is the Fibonacci sequence starting at 0.

The value of $3^\bullet \pmod{32}$ repeats every 8 (because $3^4 = 81 \equiv 1 \pmod{16}$), so we're interested in $F_n \pmod 8$. Computing enough small values of the Fibonacci sequence $\pmod 4$ shows us

$$0, 1, 1, 2, 3, 5, \quad 0, 5, 5, 2, 7, 1, \quad 0, 1, \dots.$$

The repetition of 0 and 1 implies that this sequence will repeat with period 12. Thus,

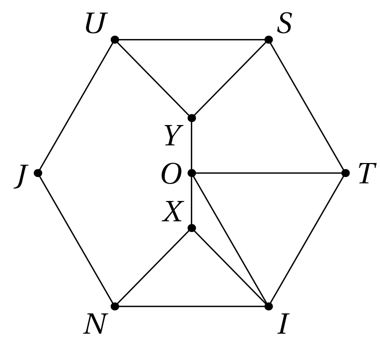
$$b_{99} = 3^{F_{99}} \equiv 3^{F_{99} \pmod 8} \equiv 3^{F_{99} \pmod{12} \pmod 8} = 3^{F_3 \pmod 8} = 3^2 = 9 \pmod{32}.$$

So $\boxed{9}$ is our answer. ∎

Proposed by Nir Elber.
Category: NT
Remark: I expect contestants to avoid the Fibonacci machinery and just compute small values of $y_\bullet$ directly.

Problem 27. Points X and Y lie in the interior of regular hexagon $JUSTIN$ such that $IX = NX = UY = SY = XY = 1$. If $TX^2 = \frac{a+b\sqrt{c}}{d}$, where a, b, c, d are positive integers, c is not divisible by the square of a prime, and no prime divides all three of a, b, d, compute $1000a + 100b + 10c + d$.

Solution. Let O be the center of $JUSTIN$. Note that since X, Y both lie on the line which perpendicularly bisects US and IN, they are collinear with O; in fact O is the midpoint of XY due to symmetry.



Now TOI is an equilateral triangle. Furthermore, $TO \parallel US$, and thus $TO \perp XY$ since XY is the perpendicular bisector of US and IN. Because of this, $\angle IOX = \angle TOX - \angle TOI = 90° - 60° = 30°$. Consider now the obtuse triangle OXI, and let P be the foot of the altitude from X to side OI. Since OXP is a 30-60-90 triangle, $XP = \frac{OX}{2} = \frac{XY}{4} = \frac{1}{4}$, and $OP = \frac{OX\sqrt{3}}{2} = \frac{\sqrt{3}}{4}$. Then $IP = \sqrt{IX^2 - XP^2} = \sqrt{1 - \frac{1}{16}} = \frac{\sqrt{15}}{4}$. So

$$OI = OP + IP = \frac{\sqrt{3} + \sqrt{15}}{4}$$

and this also equals the length of TO and the side length of $JUSTIN$ since TOI is equilateral. Thus we may use the Pythagorean Theorem:

$$TX^2 = TO^2 + OX^2 = \left(\frac{\sqrt{3} + \sqrt{15}}{4}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{3 + 15 + 2\sqrt{45}}{16} + \frac{4}{16} = \frac{22 + 6\sqrt{5}}{16} = \frac{11 + 3\sqrt{5}}{8}$$

so we answer $\boxed{11358}$. ■

Proposed by Matthew Kroesche.
Category: Geometry

Problem 28. Given that x and y are positive integers that satisfy $2x^2 + 8x = y^2 + 4y - 5$, what is the minimum value of $x + y$?

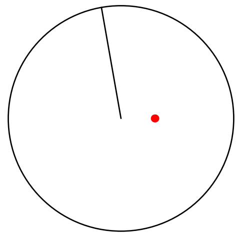
Solution. Adding 8 to both sides, we get $2x^2 + 8x + 8 = y^2 + 4y + 3$. Factoring both sides gives $2(x+2)^2 = (y+1)(y+3)$. Since the left side is even, $y+1$ and $y+3$ must be too, meaning that y must be odd. Since there are at least 2 factors of 2 on the right side, $x+2$ must be even as well, meaning that x must be even. It is also important to note that the value on the left hand side can be expressed in the form $a^2 - 1$ (plug in $a = y+2$ to see this). Testing even values of x upwards eventually gives $x = 10$, which would force $y = 15$. It is clear that increasing x among all of the positive integers would also increase the respective value of y, and therefore minimizing x would also minimize y and would therefore minimize $x + y$, and so $10 + 15 = \boxed{25}$ is the answer.

(This can also be solved using the usual Diophantine solving methods or by immediately factoring the given equations, but this seems to be the fastest solution). ■

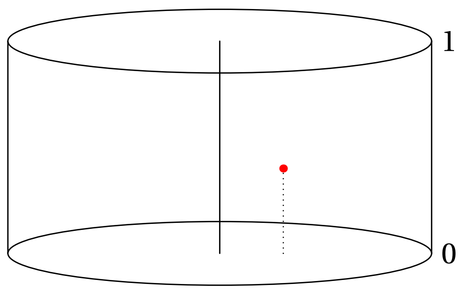
Proposed by Josiah Kiok.
Category: Algebra, NT

Problem 29. Anna throws a dart at a circular dartboard with radius 1, and Banana guesses a real distance between 0 and 1. Assuming Anna hits the dartboard randomly and Banana guesses randomly, the probability Anna's dart is closer to the center of the board than Banana's guess is the reduced fraction $\frac{m}{n}$. Compute $100m+n$.

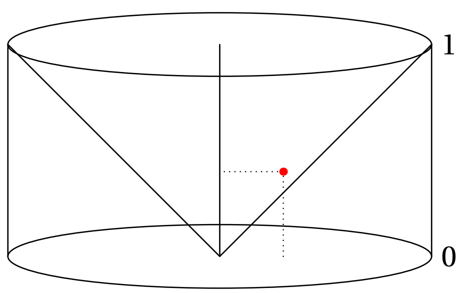
Solution. Use geometric probability. Ana's dart hits the dartboard randomly, so we can visualize its position as its position on the dartboard.



Banana's guess is orthogonal to Ana's dart and random over $[0,1]$, so we add a vertical dimension. It follows the combination of Ana's dart and Banana's guess can be visualized as a point in a cylinder with radius and height 1.



Roughly speaking, Ana's dart chooses a point on the bottom circle, and Banana's guess chooses the height. It follows that Anna's dart is closer to the center of the board than Banana's guess if and only if the generated point is closer to the vertical axis than its height. But this describes a cone!



Thus, the probability Anna's dart is closer than Banana's guess is the probability that a randomly chosen point inside of the cylinder lives inside the cone. If we let the height $h=1$ and B be the area of the base, we see that this volume ratio is

$$\frac{\frac{1}{3}Bh}{Bh}=\frac{1}{3},$$

which is what we wanted. So we answer $\boxed{103}$. ■

Proposed by Nir Elber.
Category: Combo

Problem 30. A,B,C,D are points on a circle in that order such that AD is a diameter of the circle. Let E be a point on BD such that $ECD = 45°$. If $AD = 300$, $CD = 84$, and $AB = 180$, Find DE.

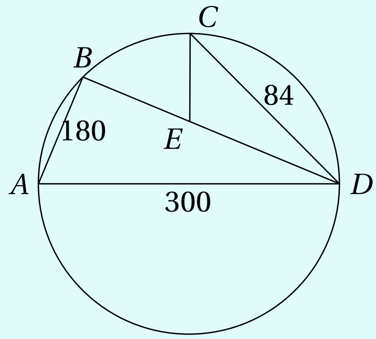


Diagram not to scale.

Solution. Let F be the intersection of AC and BD. By Pythagorean Theorem we have that $AC = 288$ and $BD = 240$. Let $BF = x$. $CF = \frac{7}{15}x$ by similar triangles, and $DF = 240-x$. Also, $AF = \frac{15}{7}(240-x)$ (also by similar triangles). Adding $AF + CF = 288$ which means that $x = 135$. Thus, $DF = 105$, CE bisects FCD, and $CF = 63$, so $DE = \frac{84}{63+84}105 = \boxed{60}$ by angle bisector theorem. ■

Solution. We find $BC = 180$ by memorizing trig values or Ptolemy:

$$180 \cdot 84 + 300 \cdot BC = 240 \cdot 288$$

Thus DE bisects ADC, and BE bisects ACD, so E is the incenter of ACD. Thus we are done by the incenter/excenter lemma, since $BE = BC = BA = 180$ and $BD = 240$.
Note: The numbers in the diagram actually do still preserve the original conditions, just with a different Pythagorean triple and double-angle. $CD = 119$, $AB = 65$, and $AD = 169$. That means that $ED = 91$. ■

Proposed by Ethan Liu.
Category: Geometry

Target Round

Problem 1. The FitnessGram PACER Test is a multistage aerobic capacity test that progressively gets more difficult as it continues. In particular, completing the first level requires runners to complete 7 laps, completing levels 2 and 3 requires runners to complete 8 laps each, completing levels 4 and 5 requires runners to complete 9 laps each, and the pattern continues with the number of laps increasing by one every two levels. How many total laps must be completed to complete the first 11 levels?

Solution. From the audio clip, level 11 is completed upon completing lap 106. However, the pattern in the real PACER test breaks down at level 10, which actually lasts 11 laps instead of the 12 laps in the problem. Hence, the answer to this problem is $\boxed{107}$ laps. ■

Proposed by Jeffrey Huang.
Category: Algebra

Problem 2. Let $ABCD$ be a square of side length 60, and let M and N be the midpoint of BC and CD respectively. Let the square $ABCD$ be folded along the lines AM, AN, and MN so that a tetrahedron is formed. Find the surface area of the tetrahedron.

Solution. Since the area remains constant, the surface area of the tetrahedron is the same as the area of the square, or $\boxed{3600}$. ■

Proposed by Joshua Pate.
Category: Geo

Problem 3. Pierce is doing a marathon play of a very difficult video game, which consists of a total of 26 levels. At the beginning, Pierce can complete two levels per day. However, due to fatigue and sleep deprivation, every day his rate of level completion will decrease by 10%. (Hence, for his second day he will only be able to complete 1.8 levels, 1.62 for the third, and so on.) What is the number of whole days it will take Pierce to complete half of the levels?

Solution. We see that $1+0.9+...+0.9^n = (1-0.9^{n+1})/0.1$. Therefore, by the $n+1th$ day Pierce will have completed $20(1-0.9^{n+1})$ levels. We want to find the lowest $n+1$ such that this is at least 13, which is equivalent to finding $1-0.9^{n+1} \geq 0.65$ or $0.9^{n+1} \leq 0.35$. Listing out powers of 9 gives that $n+1 = 10$, or that it will take $\boxed{10}$ days for Pierce to finish. ■

Proposed by Pierce Lai.
Category: Algebra
Remark: Please do not pull 10 all-nighters in a row.

Problem 4. Eight congruent spheres are arranged so that each is tangent to three others, and the centers of the spheres form a cube. Let there be two spheres, one internally tangent to the original eight, one externally tangent. Suppose the ratio of the radius of the larger sphere to the radius of the smaller sphere is $m+\sqrt{n}$, where m and n are integers. Then what is $100m+n$?

Solution. WLOG suppose the eight spheres all have radius 1, and the cube thus has side length 2. By symmetry, the centers of both the new spheres should be at the center of the cube. The distance from the center to any vertex of the cube is $\sqrt{3}$. Thus, the sphere centered at the origin that is externally tangent to these spheres has radius $\sqrt{3}-1$, and the one that is internally tangent to them has radius $\sqrt{3}+1$. Thus the ratio of the radii is $\frac{\sqrt{3}+1}{\sqrt{3}-1} = \frac{(\sqrt{3}+1)^2}{2} = \frac{4+2\sqrt{3}}{2} = 2+\sqrt{3}$. The answer is $\boxed{203}$. ■

Proposed by Joshua Pate.
Category: Geo

Problem 5. Eli has very interesting musical tastes. His playlist has N songs, three of which are by an obscure musician named Yobtu Ollaf, and the remaining $N-3$ of which are by other artists. When Eli shuffles his playlist and listens to all N songs on it in a random order, the probability that he hears all three Yobtu Ollaf songs in a row (regardless of what order they are in) is $\frac{1}{N}$. Compute N.

Solution. The number of ways to choose which three songs in the random order are Yobtu Ollaf songs is $\binom{N}{3} = \frac{N(N-1)(N-2)}{6}$, and the number of ways to position the three songs such that they play consecutively is $N-2$. So we have

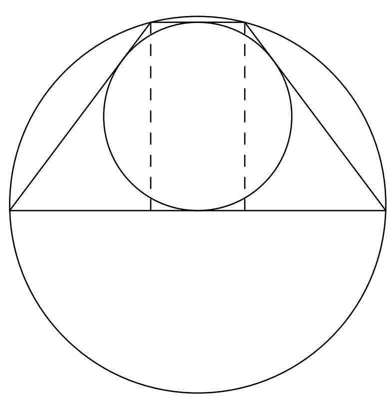
$$\frac{N-2}{\binom{N}{3}} = \frac{6}{N(N-1)} = \frac{1}{N}$$

Thus $N-1=6$, so $N = \boxed{7}$. ■

Proposed by Matthew Kroesche.
Category: Combo

Problem 6. An isosceles trapezoid has its bases of length 2 and 8, and both its legs of length 5. Alex draws a circle inscribed in the trapezoid, meaning it lies inside the trapezoid and is tangent to all four of its sides. Olivier draws a circle circumscribed about the trapezoid, meaning it passes through all four of the trapezoid's vertices. If the ratio of the area of Olivier's circle to the area of Alex's circle is written in reduced form as $\frac{m}{n}$, compute $100m+n$.

Solution. Consider the diagram below.



First, if we drop the altitudes from the endpoints of the shorter base down to the longer base, we see that they have length $\sqrt{5^2-3^2} = 4$. Thus, since the inscribed circle is tangent to both bases, its diameter must be parallel to these altitudes and thus have length 4. So it has radius 2 and area 4π.

There are a few ways to find the radius of the circumscribed circle. One is to just consider the triangle formed by three of the trapezoid's vertices (ignore one of the vertices of the shorter base) which has sides of length 5, 8, and $\sqrt{5^2+4^2} = \sqrt{41}$; and its area is $\frac{1}{2}\cdot 8\cdot 4 = 16$. Thus its circumradius is $\frac{5\cdot 8\cdot\sqrt{41}}{4\cdot 16} = \frac{5\sqrt{41}}{8}$. If one does not know the circumradius formula, it is also very reasonable to coordinate bash; consider making the midpoint of the longer base the origin and putting the vertices at $(\pm 4, 0)$ and $(\pm 1, 4)$. Then the circumcenter clearly lies on the line $x = 0$, and by forcing its distance from both vertices in the right half-plane to be the same, we find that the circumcenter has coordinates $(0, \frac{1}{8})$ – and the result follows.

Then the area of the circumscribed circle is $\frac{1025\pi}{64}$, so the ratio of the areas is $\frac{1025}{256}$. Then the answer is $100m+n = \boxed{102756}$. ■

Proposed by Matthew Kroesche.
Category: Geometry
Remark: This is an example of what is called a *bicentric quadrilateral*, i.e. one that has both a circumscribed and an inscribed circle.

Problem 7. Compute the largest prime factor of $20! \times 21!$.

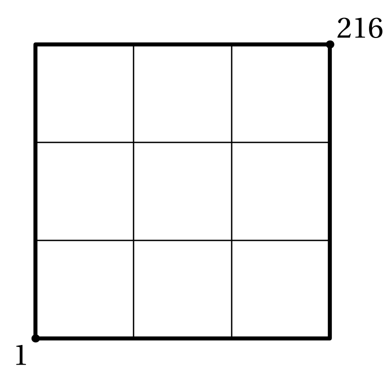
Solution. Because $n!$ is the product of all positive integers less than or equal to n, we claim the largest prime factor of $n!$ is the largest prime less than or equal to n. Indeed, this largest prime certainly divides into $n!$ because it is one of the factors, and $p \mid n!$ implies p divides some positive integer less than or equal to n, so $p \le n$.

So the largest prime factor of 20! is 19, and the largest prime factor of 21! is also 19, so the largest prime factor of $20! \times 21!$ is still $\boxed{19}$. ■

Proposed by Nir Elber.
Category: NT

Problem 8. Call a sequence *nervous* if the quotient between each term and its previous is either 2, $\frac{1}{2}$, or 3. Compute the number of nervous sequences made up of distinct positive integers with fewer than 4 digits, which start with 256 and end with 216.

Solution. This is an evil problem. What makes this possible is the observation that we are not allowed to divide by 3, which means that v_3 acts as a counter moving ever forwards. With this in mind, place the problem in a grid, where v_3 moves horizontally and v_2 moves vertically.



Realize that this grid is incomplete (256 is not even present), but it will serve as a nice visual aid. We are attempting to walk from 256, which is very high up on the leftmost column, to 216, which is the upper-right corner, in a way that does not overlap with itself.

The key claim is that the sequence is entirely defined by the first integer we choose with the number of powers of 3 equal to $1, 2, 3$. Indeed, given these integers, we can construct the corresponding sequence. This is best seen on the grid, for while we may move up and down at will (ensuring we do not overlap), we cannot go left ever, so the first integer we choose with v_3 equal to $1, 2, 3$ correspond on the grid to the first point we hit in each column. However, this is also choosing the rightward edges in our blockwalk. Then given the rightward edges, we are forced to connect the dots vertically (note no overlaps are permitted), determining the rest of our walk, which determines the sequence.

Of course, we can just manually compute the possibilities for the first integer we choose with v_3 equal to $1, 2, 3$. The other constraints at play are the fewer than four digits (i.e., < 1000) and the fact that our only other prime allowed is 2 (we can only multiply/divide by 2). Note that the power of 2 constraint means that we essentially want the number of powers of 2 less than $\frac{1000}{3^v}$ for $v = 1, 2, 3$.

1. With v_3 equal to 1, the smallest we are allowed is $3 \cdot 2^0$ of course. The largest is $3 \cdot 2^8$ because $3 \cdot 2^8 < 1000$ is equivalent to $2^8 = 256 < \frac{1000}{3}$. So we have 9 options.

2. With v_3 equal to 2, the smallest we are allowed is $3^2 \cdot 2^0$ of course. The largest is $3^2 \cdot 2^6$ because $3^2 \cdot 2^6 < 1000$ is equivalent to $2^6 = 64 < \frac{1000}{9}$. So we have 7 options.

3. With v_3 equal to 3, the smallest we are allowed is $3^3 \cdot 2^0$ of course. The largest is $3^3 \cdot 2^5$ because $3^3 \cdot 2^4 < 1000$ is equivalent to $2^5 = 32 < \frac{1000}{27}$. So we have 6 options.

This totals to $9 \cdot 7 \cdot 6 = 9 \cdot 42 = \boxed{378}$. ■

Proposed by Nir Elber.

Category: NT, Combo

Remark: The blockwalk alone would be a medium-level question. This problem is just evil. I personally would be uncomfortable with this placed in the actual written test because of its length combined with the difficulty, though perhaps the key claim is not as hard as I think it is. I'll wait for external input before I change the numbers to accomodate for difficulty.

Team Round

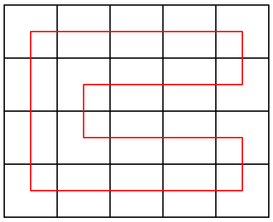
Problem 1. Ultrasonic Josh is chasing a supersonic goose. The goose flies away from Josh at 600 m/s, and Josh runs at 3000 m/s. When a goose is flying directly away from him, it takes 10 seconds for Josh to catch it. How far did the goose travel in the time Josh was chasing it?

Solution. The goose flies at 600 m/s and the flying time is 10 seconds, so the answer is $600 * 10 = \boxed{6000}$ meters. ■

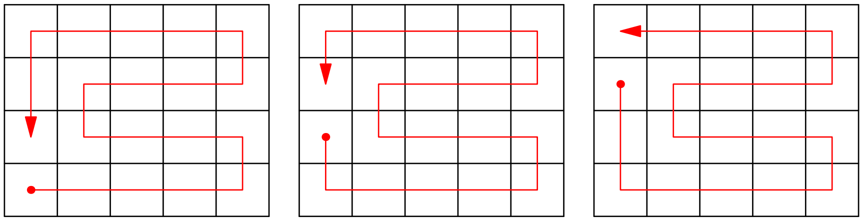
Proposed by Pierce Lai.
Category: Algebra

Problem 2. Jerry the ant drills into a $3 \times 4 \times 5$ rectangular prism made of unit cubes. He proceeds to drill from cube to adjacent cube, until he can no longer drill into an untouched cube. How many cubes can he drill into?

Solution. Consider some $1 \times 4 \times 5$ cross-section. Observe that we have the following cycle going through all cubes.



This cycle means that Jerry can drill through all 60 cubes. Indeed, Jerry can start in the bottom-left corner, move along of the first layer, follow the cycle to get all cubes in this layer, then transition to the next layer, follow the cycle again until done with the second layer, and do the same for the last layer.



Thus, Jerry can go through all $\boxed{60}$ cubes. ■

Proposed by Joshua Pate.
Category: Combo

Problem 3. What is the sum of the coefficients of $(x-2y+3z)^{10}$ when expanded?

Solution. The sum can be obtained by setting $x = y = z = 1$, so the answer is $2^{10} = \boxed{1024}$. ■

Proposed by Pierce Lai.
Category: Algebra

Problem 4. Compute the number of ordered pairs (a,b) of integer solutions to $a^2 - b^2 = 2021$.

Solution. Factor both sides like

$$(a+b)(a-b) = 2021 = 1 \cdot 2021 = 43 \cdot 47.$$

Letting $s = a+b$ and $d = a-b$, we see that $sd = 2021$, so (say) s must be an integer divisor of 2021. As factored above, this leaves us with $s \in \{\pm 1, \pm 2021, \pm 43, \pm 47\}$ as our only options. In each of these cases, we claim that we can retrieve a unique pair of integers (a,b) corresponding.

Indeed, we can solve $d = 2021/s$ is uniquely determined, and we'll see that both d and s will end up being odd. From here, we can solve $s = a+b$ and $d = a-b$ for a and b by

$$a = \frac{s+d}{2}, \qquad b = \frac{s-d}{2}.$$

This uniquely determines our a and b from s and d. Further, because both s and d are odd, $s \pm d$ will be divisible by 2, so this pair (a,b) will always have integers.

It follows that we have a unique pair (a,b) for each value of s. This gives $\boxed{8}$ total pairs. ■

Proposed by Joshua Pate.
Category: Algebra, NT

Problem 5. Theo has less than 1000 followers on InstaPix. If you divide the number of followers he has by 7, 11, and 13, you get remainders of 4, 5, and 10, respectively. How many followers does Theo have?

Solution. This is a somewhat standard Chinese Remainder Theorem problem. We want to find $0 \le x < 1000$ satisfying

$$x \equiv \begin{cases} 4 \pmod{7}, \\ 5 \pmod{11}, \\ 10 \pmod{13}. \end{cases}$$

We begin by solving

$$x \equiv \begin{cases} 4 \pmod{7}, \\ 5 \pmod{11}. \end{cases}$$

We'll add in $10 \pmod{13}$ later. Note $x \equiv 4 \pmod{7}$ means $x = 4 + 7a$ for some integer a. Combining this with the other condition, we see

$$4 + 7a \equiv 5 \pmod{11}.$$

Because $7 \cdot 8 = 55 \equiv 1 \pmod{11}$, we see $a \equiv 8 \cdot (5-4) \equiv 8 \pmod{11}$. It follows $x \equiv 4 + 7 \cdot 8 = 60 \pmod{77}$.

It remains to solve

$$x \equiv \begin{cases} 60 \pmod{77}, \\ 10 \pmod{13}. \end{cases}$$

Note $x \equiv 60 \pmod{77}$ implies $x = 60 + 77b$ for some integer b. Combining this with the other condition, we see

$$60 + 77b \equiv 10 \pmod{13}.$$

However, $77 \equiv -1 \pmod{13}$, so we see $b \equiv -1 \cdot (10 - 60) \equiv -2 \pmod{13}$. It follows $x \equiv 60 + 77 \cdot (-2) = 60 - 154 = -94 \pmod{13 \cdot 77}$, which means

$$x \equiv 907 \pmod{1001}.$$

It follows that $x = \boxed{907}$. ■

Proposed by Joshua Pate.
Category: NT

Problem 6. Let A, B, and C be adjacent squares with side lengths 8, 12, and 8, respectively, each with two vertices on the horizontal line ℓ. Let O be the center of the circle passing through the 4 vertices among the three squares that do not touch the line ℓ. Let ℓ intersect the circle at points A and B. What is the value of AB^2?

Solution. We use coordinates. Let $(0,0)$ be the center of the top edge of the center square (assuming ℓ is the bottom edge). Then the circumcenter O will be the intersection of the line CD with the y-axis, where C is the point $(6,0)$ and D is the point $(14,-4)$. The midpoint of CD is $(10,-2)$, and the perpendicular bisector of CD is the line $y = 2x - 22$. Hence, the center O is $(0,-22)$. Hence, if r is the circumradius, we get $r^2 = 22^2 + 6^2 = 520$. Half the distance from O to ℓ is 10, so from the Pythagorean Theorem, $(AB/2)^2 = 520 - 100 = 420$.

In conclusion, $AB^2 = \boxed{1680}$. ■

Proposed by Jeffrey Huang.
Category: Geometry

Problem 7. Timmy plays a game with numbers, starting with the single number 1. Every turn, he flips a fair coin ($\frac{1}{2}$ probability of each result). If the coin flips heads, the number is multiplied by 3, otherwise, it is multiplied by $\frac{1}{3}$. If his expected score after 4 turns is $\frac{m}{n}$ when written as a reduced fraction, find $100m + n$.

Solution. Expectation is actually also multiplicative! The expected value is exactly $(\frac{1}{2}(3 + \frac{1}{3}))^4$ (this can be proved with induction or just "common sense"), which is $\frac{625}{81}$. So we answer $\boxed{62581}$. ■

Proposed by Ethan Liu.
Category: Combo

Problem 8. Let A,B,C be points on circle ω in the coordinate plane centered at the origin such that B, C have positive x-coordinates, A does not, and the y-coordinates of A, B, C are 300, 260, 91 respectively. Let A', B', C' be the feet of the altitudes from A, B, C to the x-axis. Let E and F be points inside the circle such that $AEA' \sim BEB'$ and $BFB' \sim CFC'$. If the radius r of the circle is 325, then $\frac{EF}{r} = \frac{m}{n}$, where the fraction $\frac{m}{n}$ is reduced. Find $100m + n$.

Solution. We can find all x-values: $x_A = -125$, $x_B = 195$, and $x_C = 312$. Through that, we can use side ratios to find $B'E = 320(\frac{26}{56})$ and $B'F = 117(\frac{10}{27})$. Thus $EF = 260(\frac{19}{21})$. $\frac{EF}{r} = \frac{76}{105}$. So we write our answer as $\boxed{7705}$. ■

Proposed by Ethan Liu.
Category: Geometry
Remark: I didn't like $EF = 260(\frac{19}{21})$, so I made the answer a little nicer on the eyes. Moved to medium because its just Pythagorean Theorem plus a little length stuff.

Problem 9. Leon is thinking of a quadratic polynomial $f(x)$ with integer coefficients with the property that $0 < f(1) < f(2) < f(0)$. He defines the function $g(x) = xf(x)f(f(x))$ and calculates $g(1) = 511$. Given this information, compute $g(3)$.

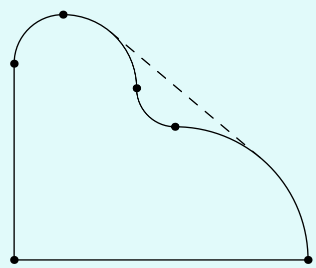
Solution. First, we have $f(1)f(f(1)) = 511$, so $f(1)$ divides $511 = 7 \cdot 73$. Now $f(1) > 0$, and so $f(1)$ is one of $1, 7, 73, 511$. If $f(1) = 1$, then $g(1) = 1 \cdot 1 \cdot 1 = 1 \neq 511$. If $f(1) = 73$, then $f(f(1)) = f(73) = 7$, which is a contradiction since the parabola opens upward and has its vertex between 0 and 2. Similarly, if $f(1) = 511$, then $f(511) = 1$ which is a contradiction since it's too low. So it must be that $f(1) = 7$ and $f(7) = 73$.

Next, let $f(x) = ax^2 + bx + c$. Then $a + b + c = 7$ and $49a + 7b + c = 73$. Rearranging, $a + b = 7 - c$ and $7a + b = \frac{73-c}{7}$. Thus we can solve this to get $a = \frac{c+4}{7}$ and $b = \frac{45-8c}{7}$. Now we have $c = f(0) > f(1) = 7$. We also have $f(2) > f(1) = 7$, so $8a+4b+c = \frac{8(c+4)}{7} + \frac{4(45-8c)}{7} + c = \frac{212-15c}{7} > 7$. Thus $212-15c > 49$ and so $15c < 163$. Thus $c < \frac{163}{15} < 11$. Finally $c \equiv 3 \pmod 7$, or else a, b are not integers. So it must be that $c = 10$. Then $a = 2$ and $b = -5$. Thus $f(x) = 2x^2 - 5x + 10$.

Finally, we compute $f(3) = 2(9) - 5(3) + 10 = 13$, and $f(f(3)) = f(13) = 2(169) - 5(13) + 10 = 283$. Thus $g(3) = 3f(3)f(f(3)) = 3(13)(283) = \boxed{11037}$. ■

Proposed by Matthew Kroesche.
Category: Algebra, NT

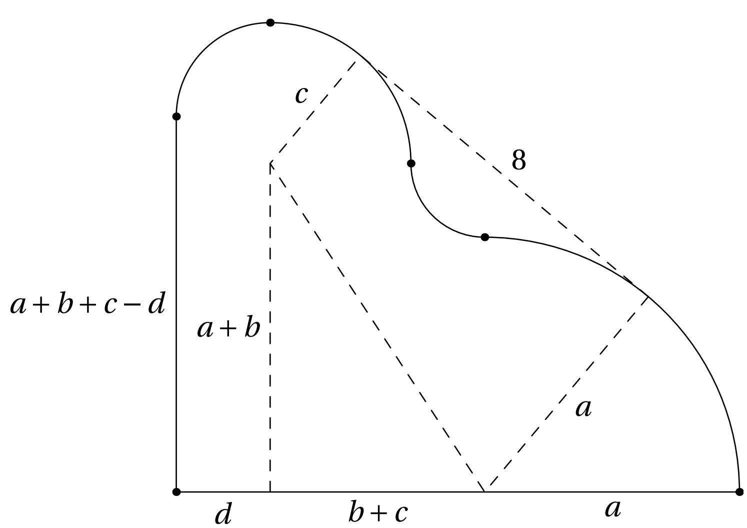
Problem 10. Andy is measuring the grand piano shown below, which consists of four quarter-circle arcs, each of which has rational radius, and two straight line segments. First, he finds that its perimeter is $20 + 6\pi$. Then, he holds his measuring stick in the unique way such that it touches the curved part of the piano twice, and finds that the distance between the two points where it touches is 8. If the area of the piano is $p + q\pi$ where p is an integer and q is rational, compute p.



Solution. Going counterclockwise around the piano, label the lengths of the arcs as a, b, c, d. Then we see that the vertical line segment has length $a+b+c-d$ and the horizontal segment has length $a+b+c+d$. Meanwhile the four circular arcs have total length $\frac{\pi}{2}(a+b+c+d)$. So

$$2(a+b+c) + \frac{\pi}{2}(a+b+c+d) = 20 + 6\pi$$

Therefore $a+b+c+d = 12$ and $a+b+c = 10$, so $d = 2$. We can also make use of the tangency measurement by connecting the centers of the first and third circles, and drawing the radii to the points of tangency to construct a trapezoid:



The bases of this trapezoid have length a and c. Its longer leg has length $\sqrt{(b+c)^2+(a+b)^2}$ due to the Pythagorean Theorem, and its shorter leg has length 8. Collapsing the trapezoid, we can turn it into a right triangle with legs of length 8 and $a-c$, and hypotenuse $\sqrt{(b+c)^2+(a+b)^2}$. Thus, using the Pythagorean Theorem a second time,

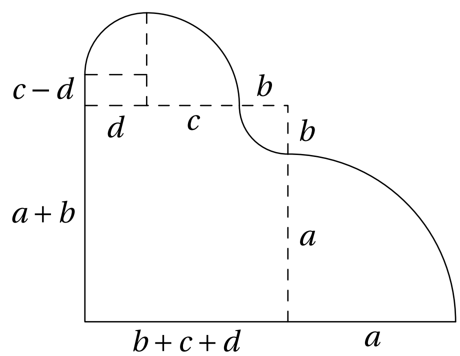
$$64+(a-c)^2 = (b+c)^2+(a+b)^2$$

$$64+a^2-2ac+c^2 = b^2+2bc+c^2+a^2+2ab+b^2$$

$$64 = 2b^2+2bc+2ab+2ac$$

$$(a+b)(b+c) = 32$$

Now we draw some different lines to calculate the area of the piano.



From this picture, we see that the area is

$$(a+b)(b+c+d)+d(c-d)+\frac{\pi}{4}(a^2-b^2+c^2+d^2)$$

Thus

$$\begin{aligned} p &= (a+b)(b+c+d)+d(c-d) \\ &= (a+b)(b+c)+d(a+b+c-d) \\ &= 32+2(10-2) \\ &= 32+16 \\ &= \boxed{48} \end{aligned}$$

■

Proposed by Matthew Kroesche.
Category: Geometry, Algebra

Countdown Round

Problem 1. For a round of golf, the score of a player equals the number of hits the player takes to hit the ball into the hole minus the par constant for that round. Hence, lower scores are considered better. In a 9-hole game consisting of 3 holes with par 3, 3 holes with par 4, and 3 holes with par 5, the only hole in which Gary scores under par is a hole-in-one. If Gary takes at most 10 hits to finish each hole, what is the best possible average score that Gary can get in this 9-hole game? Express your answer as a common fraction.

Solution. Other than the hole-in-one, every other hole should be scored as par, or 0. The hole-in-one that optimizes Gary's score is when par is 5. Hence, the best possible average score is $\boxed{-\frac{4}{9}}$. ■

Proposed by Jeffrey Huang.
Category: Algebra

Problem 2. Justin has a large number of sausages which he is consuming. However, the smell of sausages attracts large numbers of raccoons, which he must fend off lest they overwhelm him and consume his sausages. If Justin can normally consume 1 sausage per second, but every second one raccoon appears and lowers his consumption speed by 0.04 sausages a second, then what is the maximum number of whole sausages Justin can consume before he is too busy fending off raccoons to eat any more? (Assume the first raccoon appears precisely one second after Justin begins eating.)

Solution. We see that the answer is just an arithmetic series $(1+.96+.92+.88+\ldots+.04)$, which has 25 terms. Hence, the answer is $25 * .52 = \boxed{13}$ sausages. ■

Proposed by Pierce Lai.
Category: Algebra

Problem 3. What is the largest positive integer with distinct odd digits that is divisible by 9?

Solution. There are only 5 odd digits: 1, 3, 5, 7, 9, with a sum of 25. Hence, no 5-digit positive integer is divisible by 9. We can form a valid 4-digit number by removing the digit 7. The largest such positive integer is $\boxed{9531}$. ■

Proposed by Jeffrey Huang.
Category: NT

Problem 4. Eddie fully expands $(1+x)^8$ and randomly picks a coefficient. What is the expected value of the coefficient? Express your answer as a common fraction.

Solution. Let $n = 8$. If Eddie picks the monomial $\binom{n}{k}x^k$, then the value of the coefficient is $\binom{n}{k}$ and occurs with probability $\frac{1}{n+1}$. So our expected value is

$$\sum_{k=0}^{n} \frac{1}{n+1}\binom{n}{k} = \frac{1}{n+1}\cdot(1+1)^n = \frac{2^n}{n+1}.$$

It follows our answer is $\frac{2^8}{9} = \boxed{\frac{256}{9}}$. ■

Proposed by Nir Elber.
Category: Algebra

Problem 5. Madeline is playing a video game, but she dies a lot. Each time she dies, she has a 1/3 chance of falling to her death, a 1/2 chance of being speared by cacti, and a 1/6 chance of being crushed. If she dies 3 times, what is the chance that she will die in a different way each time? Express your answer as a common fraction.

Solution. The probability is $6 \times \frac{1}{3} \times \frac{1}{2} \times \frac{1}{6} = \boxed{\frac{1}{6}}$. ■

Proposed by Pierce Lai.
Category: Combo

Problem 6. When Kate divides any positive integer g (that is divisible by n) by n, she can guarantee that she knows exactly what the last two digits of the quotient are. How many positive integers less than or equal to 100 can n be?

Solution. The problem essentially asks how many positive integers n have an inverse modulo 100. The answer is $\varphi(100) = 100(1 - \frac{1}{2})(1 - \frac{1}{5}) = \boxed{40}$. ■

Proposed by Jeffrey Huang.
Category: NT

Problem 7. How many integer solutions are there to $x^2 + 2y^2 = 81$?

Solution. Notice that $2y^2 \le 81$ means $y^2 \le 40$, so $|y| < 6$. Further, note that all squares are 0 or 1 $\pmod 4$, so if $|y|$ is odd, then $y^2 \equiv 1 \pmod 4$, so $x^2 = 81 - 2y^2 \equiv 3 \pmod 4$, which is impossible. So y is even. Thus, there aren't that many cases to work through.

- If $|y| = 0$, then $x^2 = 81$, so $x = \pm 9$. This gives 2 solutions.
- If $|y| = 2$, then $x^2 = 73$, which gives no solutions.
- If $|y| = 4$, then $x^2 = 49$, so $x = \pm 7$. This gives 4 solutions.
- If $|y| = 6$, then $x^2 = 9$, so $x = \pm 3$. This gives 4 solutions.

In total, we have $\boxed{10}$ solutions. ■

Proposed by Joshua Pate.
Category: NT

Problem 8. What is the greatest positive integer b for which the base-10 representation of 2021_b contains at most 3 digits?

Solution. We would like for $2b^3 + 2b + 1 < 1000$. Since $2b^3 = 1024 > 1000$, we check $b \le 7$. Luckily, $2 \times 7^3 + 2 \times 7 + 1 = 686 + 15 = 701 < 1000$, so the answer is $\boxed{7}$. ■

Proposed by Jeffrey Huang.
Category: NT

Problem 9. The side length of the base $ABCD$ of a square pyramid is 6 units, and the maximum length among the remaining edges EA, EB, EC, and ED is 8 units. What is the maximum possible volume of the square pyramid, in cubic units? Express your answer in simplest radical form.

Solution. We need to find the point F on the plane $ABCD$ that minimizes the maximum distance between F and any of the vertices of the square. One can prove that this happens when F is the center of $ABCD$, which means E should lie above the center of the square. From the Pythagorean Theorem, the height of the pyramid is $\sqrt{46}$ units, so using $V = \frac{1}{3}Bh$, the volume of the square pyramid is $\boxed{12\sqrt{46}}$ cubic units. ■

Proposed by Jeffrey Huang.
Category: Geo

Problem 10. How many non-congruent scalene triangles ABC with integer side lengths are there such that $AB = 9$, $BC = 11$, and angle ABC is acute?

Solution. In order for angle ABC to be acute, we need $AC < \sqrt{9^2 + 11^2} = \sqrt{202}$. Hence, we need AC to be an integer between 3 and 14, inclusive, excluding 9 and 11 for scalene-ness. Hence, there are a total of $\boxed{10}$ triangles. ■

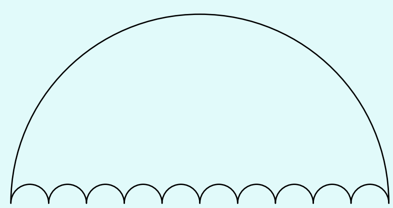
Proposed by Jeffrey Huang.
Category: Geo
Remark: too many 10s

Problem 11. Farmer Ethan has 152 ducks, 12 cats, and 520 roosters. If he can trade 12 ducks for a pig, find at least how many more ducks he needs to obtain 15 pigs.

Solution. $15 \cdot 12 - 152 = 180 - 152 = \boxed{28}$. ■

Proposed by Alex Zheng.
Category: Algebra

Problem 12. The figure shown below is a semicircle with diameter 1 where instead of having a straight line, it has 10 equal perforations, each in the shape of a semicircle. Compute its perimeter in terms of π.



Solution. Let $N = 10$ be the number of perforations. The top semicircle has perimeter $\frac{1}{2} \cdot \pi \cdot 1 = \frac{1}{2}\pi$. Each perforation has diameter $\frac{1}{N}$ and therefore perimeter $\frac{1}{2} \cdot \pi \cdot \frac{1}{N} = \frac{1}{2N}\pi$. This totals to

$$\frac{1}{2}\pi + N \cdot \frac{1}{2N}\pi = \frac{1}{2}\pi + \frac{1}{2}\pi = \pi.$$

Thus our answer is $\boxed{\pi}$. ■

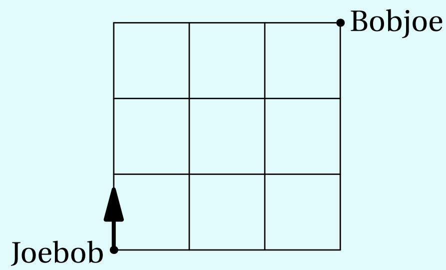
Proposed by Joshua Pate.
Category: Geo
Remark: Yell at Nir if the asymptote doesn't look right.

Problem 13. Let n be a random two-digit positive integer, let u be the sum of their digits, and let $t = n + u$. What is the probability that t is divisible by 7? Express your answer as a common fraction.

Solution. Expressing $n = 10a + b$, we would like for $11a + 2b \equiv 4a + 2b$ to be divisible by 7. Dividing by 2, we would like for $2a + b$ to be divisible by 7. Scanning across all a from 3 to 9, we have 10 of the 70 integers are divisible by 7. For $a = 1$, we need $b = 5$, and for $a = 2$, we need $b = 3$. Hence, the desired probability is $\frac{12}{90} = \boxed{\dfrac{2}{15}}$. ■

Proposed by Jeffrey Huang.
Category: Combo, NT

Problem 14. Joebob the ant is visiting his friend Bobjoe on the coordinate plane. Joebob starts at the origin facing north, and Bobjoe is waiting at the point (3,3). Every minute, Joebob can choose to either walk forward one unit, or turn left 90°. If Joebob must reach Bobjoe by the end of 10 minutes, how many possible sequences of choices are there for Joebob to reach Bobjoe in time? (Assume the sequence stops when Joebob reaches Bobjoe.)



Solution. Notice that the fastest path requires 9 minutes - if we let W represent walking forward and T represent turning, the fastest path would be WWWTTTWWW. Hence Joebob has one extra minute. Note that Joebob must only walk upwards and rightwards (as not doing so would require 2 extra minutes). Thus, Joebob can only use the minute to make an extra turn. This gives a total of 4 paths (WWWTTTWWW, WWTTTWWWTW, WTTTWWWTWW, TTTWWWTWWW) so the answer is $\boxed{4}$. ■

Proposed by Pierce Lai.
Category: Combo
Remark: Yell at Nir if you don't like the diagram.

Problem 15. A white $7 \times 10 \times 13$ prism is dipped into red paint, and then cut into $1 \times 1 \times 1$ cubes. How many $1 \times 1 \times 1$ cubes have at least one red face?

Solution. The number of cubes without any painted faces is $5 * 8 * 11 = 440$, so the answer is $910 - 440 = \boxed{470}$. ■

Proposed by Pierce Lai.
Category: Combo

Problem 16. Alex is running a fried snow stand. If each order of fried snow can come with one or two sauces (ranch, mayonnaise, chocolate, pickle brine, or barbeque), one type of topping (vanilla pellets, lettuce, or surströmming), and one, two, or three (distinct) sides (extremely short hotdog, century egg, mellified mountain dew, banana peel soylent, cordyceps fries, or dorito slushie) how many different orders of fried snow can he serve?

Solution. There are $5+10=15$ choices for the sauce, 3 choices for the topping, and $6+15+20=41$ choices for the sides, so the answer is $15*3*41=\boxed{1845}$. ■

Proposed by Pierce Lai.
Category: Combo
Remark: It's hilarious but way too much flavortext. I think the wacky flavors at the end can all stay, but maybe a couple of sentences at the beginning should be trimmed.

Problem 17. What is the remainder when 123,456,789 is divided by 99?

Solution. Add them in blocks of two digits at a time. We get $1+23+45+67+89=225$, which has a remainder of $\boxed{27}$ when divided by 99. ■

Proposed by Jeffrey Huang.
Category: NT

Problem 18. A cube is inscribed in a sphere, which is inscribed in a cube. If the outer cube has volume 216 and the inner cube has volume V, find V^2.

Solution. The outer cube has side length 6, so the sphere has radius 3, so the inner cube has a long diagonal of length 6. Thus, the inner cube has side length $2\sqrt{3}$, so it has a volume of $V=24\sqrt{3}$. Then $V^2=576\cdot 3=\boxed{1728}$. ■

Proposed by Pierce Lai.
Category: Geo

Problem 19. If a, b, and c are distinct positive integers such that $c>100$ and $\sqrt{a}+\sqrt{b}=\sqrt{c}$, what is the least possible value of c?

Solution. We are looking for the smallest possible value of c such that $\sqrt{c}=s\sqrt{t}$ in simplest radical form, where $s\geq 3$. Scanning through positive integers greater than 100, we find our desired answer of $\boxed{108}$. ■

Proposed by Jeffrey Huang.
Category: Algebra

Problem 20. Lea loves sandwiches. She can eat 10 regular sandwiches and 6 Hi-sandwiches, or 4 Chef sandwiches, 1 Hi-sandwich, and 5 regular sandwiches, or 3 Chef sandwiches and 6 Hi-sandwiches. If she eats 2 Chef sandwiches and 5 regular sandwiches, how many Hi-sandwiches can she eat?

Solution. If r represents the size of a regular sandwich, h for Hi-, and c for chef, then we have the equations $10r+6h=5r+h+4c=6h+3c$. The first two tell us that $5r+5h=4c$, and the last two tell us that $5r+c=5h$. These tell us that $3c=10r$ and $3h=5r$, so overall she can eat the equivalent of 20 regular sandwiches. After eating 2 Chef sandwiches and 5 regular sandwiches, she has the equivalent of 25/3 regular sandwiches' room left, so the answer is $25/3/(5/3)=\boxed{5}$ Hi-sandwiches. ■

Proposed by Pierce Lai.
Category: Algebra
Remark: This needs to be reworded.

Problem 21. A perfect square is a positive integer that is equal to the product of an integer and itself. What is the sum of the three-digit positive integers that are squares of perfect squares?

Solution. We are asked for the sum of the three-digit 4^{th} powers, so the answer is $256 + 625 = \boxed{781}$. ■

Proposed by Jeffrey Huang.
Category: Algebra

Problem 22. In a private jet, there are 3 distinguishable adults and 5 distinguishable kids who must sit to form 2 distinguishable rows of 4 people. How many ways are there to sit the 8 people if there must be an adult sitting in the rightmost seat of each row?

Solution. Aside from the two rightmost seats being adults, the last adult can fill in any of the 6 remaining seats. There are $3! = 6$ ways to fill the adults and $5! = 120$ ways to fill the kids. This gives a total of $\boxed{4320}$ ways to arrange the 8 people in the seats of the private jet. ■

Proposed by Jeffrey Huang.
Category: Combo

Problem 23. If $a = 2^{\sqrt[4]{a}}$ and a is a positive integer, compute the positive integer n for which $a = 2^n$.

Solution. Let $a = 2^n$. Then

$$2^n = 2^{\sqrt[4]{2^n}} = 2^{2^{n/4}}.$$

Thus $n = 2^{n/4}$. A quick guess tells us $n = \boxed{16}$ is a solution (the other solution does not give a to be an integer). ■

Proposed by Pierce Lai.
Category: Algebra

Problem 24. What is the least positive integer n such that the product $(1)(1+2)(1+2+3)(1+2+3+4)...(1+2+3+4+...+n)$ is divisible by 2021^3?

Solution. Because 2021 is odd, we do not care about the 2's in the denominators of the expressions $1+2+3+4+...+k = \frac{k(k+1)}{2}$. Because $2021 = 43 \times 47$, we need to see the value 47 appear at least 3 times, and 43 will appear 3 times earlier on. They appear for $k = 46$ and $k = 47$. The third time they appear is when $k = 2 \times 47 - 1 = \boxed{93}$. ■

Proposed by Jeffrey Huang.
Category: Algebra, NT

Problem 25. If x is a positive real number such that $x^2 = x + 1$, what is the value of $x^4 + x^3 + x^2 + x + 1$? Express your answer as a common fraction in simplest radical form.

Solution. From the quadratic formula, $x = \frac{1+\sqrt{5}}{2}$. Substitute each instance of x^2 as $x + 1$ until the desired polynomial is only linear. The linear expression is $7x + 5 = \boxed{\dfrac{17+7\sqrt{5}}{2}}$. ■

Proposed by Jeffrey Huang.
Category: Algebra

Problem 26. Donald and Joe are eating some potatoes. Donald can consume 3 potatoes a minute, while Joe can consume 5 potatoes every 2 minutes. Assuming they have unlimited stomach capacity, how long will it take for them to consume 165 potatoes total?

Solution. Joe can consume 2.5 potatoes every minute, so they can consume 5.5 potatoes each minute total. Thus, the answer is $165/5.5 = \boxed{30}$ minutes. ■

Proposed by Pierce Lai.
Category: Algebra

Problem 27. If $a\#b = a^2 + 3ab + b^2$ for all real numbers a and b, name the ordered pair of positive integers (a, b) such that $a \le b$, $a\#b = 99$, and $a + b$ is as small as possible.

Solution. The quadratic equation gives us $b = \frac{-3a \pm \sqrt{5a^2+396}}{2}$, so we search for values of a which make $5a^2 + 396$ a perfect square. Because # is commutative, we need only search for (a, b) for which $99 = a^2 + 3ab + b^2 \ge 5a^2$, or $a \le 4$. Testing, we find that only $a = 3$ works, from which we also get $b = \frac{-9\pm 21}{2}$. Since b is a positive integer, we must have $b = 6$. This gives the ordered pair $\boxed{(3, 6)}$.

Alternatively, taking modulo 3, we see that $a^2 + b^2 \equiv 0 \pmod 3$ requires $a, b \equiv 0 \pmod 3$. Because the expression is homogeneous with degree 2, we can divide the equation by 9 to get $a\#b = a^2 + 3ab + b^2 = 11$, from which $11 \ge 5a^2$ forces $a = 1$, in which $b = 2$ comes out easily. Multiplying these constants back by 3, we get our desired solution of $\boxed{(3, 6)}$. ■

Proposed by Jeffrey Huang.
Category: Algebra, NT

Problem 28. Two people are standing on top of a high place, when - to their great horror - they see Dracula rising up towards them with a jetpack. They immediately start climbing away at 4 miles per hour. If Dracula is 80 meters away, and his jetpack goes at 2 meters per second, how long in minutes will it take for Dracula to catch them? (Assume a mile is 1600 meters.)

Solution. Four miles an hour is $4 * 1600/3600 = 16/9$ meters per second, so Dracula is catching up to them at $2/9$ meters per second. Hence, it will take $80/(2/9)/60 = \boxed{6}$ minutes. ■

Proposed by Pierce Lai.
Category: Algebra

Problem 29. Let the rays AB and DC of circle O intersect outside the circle at point E. If $EA = 7$ units, $EB = 9$ units, and $EC = 8$ units, what is the length of ED, in units?

Solution. Using Power of a Point on point E with respect to circle O, we get $9 \times 16 = 8 \times (8 + x)$, or $x = \boxed{10}$ units. ■

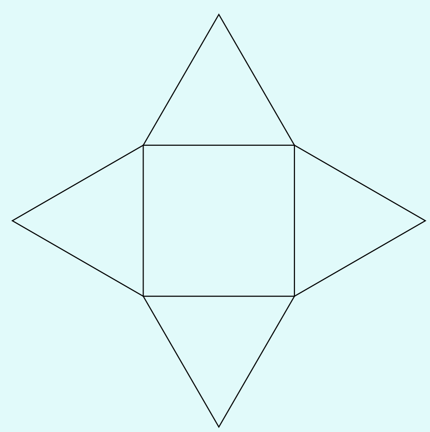
Proposed by Jeffrey Huang.
Category: Geo
Remark: too many 10s

Problem 30. Farmer Bob owns chickens, cows, and goats. Each chicken has two legs while each cow and goat have four. Cows and goats have two horns each, and every day each cow gives 7 gallons of milk per day while each goat gives 2 gallons. (Chickens do not have horns and do not give milk.) If his animals have 300 total legs, 100 horns, and produce 200 gallons of milk per day, what is the total number of animals he has?

Solution. Subtracting the total number of horns from the number of legs is twice the total number of animals, so the answer is $(300-100)/2 = \boxed{100}$ animals. (There are 50 chickens, 20 cows, and 30 goats.) ■

Proposed by Pierce Lai.
Category: Algebra

Problem 31. An octagonal prism has a base consisting of a square with 4 equilateral triangles attached, one on each side. If the base has a side length of 1 meter and the prism has a height of 2 meters, compute the volume of the prism. Express your answer in simplest radical form.



Solution. The octagon has area $1+\sqrt{3}$ square meters, so the prism has volume $\boxed{2+2\sqrt{3}}$ cubic meters. ■

Proposed by Pierce Lai.
Category: Geometry

Problem 32. What is the sum of the two smallest positive integers with exactly 18 integer factors?

Solution. Since integer factors include positive and negative integers, we want to find positive integers with 9 positive integer factors. Such positive integers must be of the form p^8 or p^2q^2 for p and q prime. The smallest such positive integers are 36 and 100, which sum to $\boxed{136}$. ■

Proposed by Jeffrey Huang.
Category: NT

Problem 33. What is the value of $\frac{142857}{111111}+\frac{428571}{444444}+\frac{285714}{222222}+\frac{857142}{888888}+\frac{571428}{555555}+\frac{714285}{777777}$? Express your answer as a common fraction.

Solution. Dividing every numerator and denominator by 999999, we get $\frac{1/7}{1/9}+\frac{3/7}{4/9}+\frac{2/7}{2/9}+\frac{6/7}{8/9}+\frac{4/7}{5/9}+\frac{5/7}{7/9}=\frac{9}{7}(1+\frac{3}{4}+1+\frac{3}{4}+\frac{4}{5}+\frac{5}{7})=\frac{9}{7}\times\frac{351}{70}=\boxed{\frac{3159}{490}}$. ■

Proposed by Jeffrey Huang.
Category: Arithmetic, Algebra, NT
Remark: The hardest part of this problem is the computation.

Problem 34. A Florida man starts at the origin. In the first hour he walks 1 unit upwards, then 1 unit left, then 1 unit down, then 2 units right. Then in the second hour he walks 2 units upward, 2 units left, 2 units down, and then 3 units right. For the third hour, he walks 3 units upward, 3 units left, 3 units down, and 4 units right, and so on. If he does this for 100 hours, how many units will the Florida man be from the origin at the end of the 100th hour?

Solution. Each time, the Florida man just essentially moves one unit to the right, so the answer is $\boxed{100}$. ■

Proposed by Pierce Lai.
Category: Geo

Problem 35. Philip flips a weighted coin that flips heads with probability f. If the expected number of Philip's flips which flip heads in 44 flips is $5f+5$, find f. Express your answer as a common fraction.

Solution. The expected number of heads is $44f$. It follows that $44f=5f+5$, which gives $f=\boxed{\frac{5}{39}}$. ■

Proposed by Nir Elber.
Category: Combo
Remark: Not enough f.

Problem 36. Find $\frac{1}{2}(2^5-6^2)+18$.

Solution. $\frac{1}{2}(2^5-6^2)+18=\frac{1}{2}(-4)+18=-2+18=\boxed{16}$. ■

Proposed by Alex Zheng.
Category: Algebra

Problem 37. What is the positive difference between the maximum and minimum possible values of the expression $6-5\times 4^3$ if any number of parentheses may be added to change the order of operations?

Solution. The maximum comes by not allowing 5 to combine with anything else before subtracting. This gives a maximum of 64. The minimum comes from multiplying 5×4 first before cubing. The minimum is $6-8000$. The positive difference is $8000-6+64=\boxed{8058}$. ■

Proposed by Jeffrey Huang.
Category: Arithmetic
Remark: This is very off-style. Also, have we learned nothing from our dreadful mistake three years ago?

Problem 38. While competing virtually at the Fairly Awful Relay Management League (FARML), a group of three students is trying to agree on an online platform to take the Relay round. Each student randomly (and independently of the other students) chooses to use one of three platforms: Zoob with probability $\frac{1}{2}$, Fiscord with probability $\frac{1}{3}$, and Smack with probability $\frac{1}{6}$. What is the probability that all three students use the same platform? Express your answer as a common fraction.

Solution. The probability is $\left(\frac{1}{2}\right)^3 + \left(\frac{1}{3}\right)^3 + \left(\frac{1}{6}\right)^3 = \frac{3^3+2^3+1^3}{6^3} = \frac{36}{216} = \boxed{\frac{1}{6}}$. ■

Proposed by Matthew Kroesche.
Category: Combo
Remark: OK I promise this is the last roast problem I'll write, but I couldn't resist...

Problem 39. John Johnson has an esteemed career of doing Job at Place. He did 1 Task on his first day at Job, and each day after he doubles the Tasks he did the previous day, with one more Task for style. Compute the total number of Tasks John does on his first 10 days doing Job.

Solution. We can show inductively that John does $2^k - 1$ tasks on day k. Alternatively, writing it out explicitly, he does

$$\underbrace{2(2(\cdots 2}_{(k-1)\text{ twos}}(1)+1\cdots)+1)+1 = 2^{k-1} + 2^{k-2} + \cdots + 2 + 1.$$

Then the geometric sum formula says that this is $\frac{2^k-1}{2-1} = 2^k - 1$.

Thus, we want to evaluate

$$\sum_{k=1}^{10} \left(2^k - 1\right) = \left(2^1 + 2^2 + \cdots + 2^{10}\right) - 10.$$

Again, the geometric sum formula says that this is $\frac{2^{11}-2}{2-1} - 10 = 2048 - 2 - 10 = \boxed{2036}$. ■

Proposed by Nir Elber.
Category: Algebra
Remark: There's a clean solution by just putting everything in binary.

Problem 40. What is the only digit that is not present in the decimal expansion of $\frac{1}{81}$?

Solution. Dividing $0.\overline{111111111}$ by 9, we get $0.\overline{012345679}$. The only digit not present is $\boxed{8}$. ■

Proposed by Jeffrey Huang.
Category: Algebra

Problem 41. The integer common angle of a regular u-gon is 1 degree greater than the integer common angle of a regular v-gon. What is the minimum possible value of v?

Solution. Looking at exterior angles, we want to solve the equation $\frac{360}{v} - \frac{360}{u} = 1$, and we are given that these values are both integers. In other words, we need to find two consecutive positive integers that are factors of 360 and are as large as possible. If a and b are such factors, then we must also have ab divides 360. The most optimal scenario would be to use 9 and 10, giving us $\frac{360}{v} = 10$, or $v = \boxed{36}$. ■

Proposed by Jeffrey Huang.
Category: NT, Geometry

Problem 42. 64 people are participating in a single-elimination hot potato tournament. Each match consists of a 1 v 1 hot potato game, where each player has a 1/3 chance to win, lose, or draw. Once a player loses, they are eliminated from the tournament. The tournament ends once there is only one person left who has not been eliminated. What is the expected number of matches required to determine the winner of the tournament? Express your answer as a common fraction.

Solution. Every match is expected to eliminate 2/3 people, so since 63 people need to be eliminated, the expected number of matches is $63/(2/3) = \boxed{\frac{189}{2}}$. ■

Proposed by Pierce Lai.
Category: Algebra

Problem 43. In a forest, small bushes contain 9 nuts, medium bushes contain 14 nuts, and large bushes contain 20 nuts. Tico plans to search through 6 bushes, 3 of which must be of one size, 2 of which must be a different size, and the last of which must be of the last size. What is the difference between the maximum and minimum number of nuts Tico can collect?

Solution. The extremes are attained by taking $(S, M, L) = (3,2,1),(1,2,3)$. The difference is $2 \times 20 - 2 \times 9 = \boxed{22}$ nuts. ■

Proposed by Jeffrey Huang.
Category: Algebra

Problem 44. For what positive integer base b does $323_b = 123_{10}$?

Solution. We see that $123/3 = 41$, which is a little larger than 36. Hence, we test 6, which works since $108+12+3 = 123$, so the answer is $\boxed{6}$. ■

Proposed by Pierce Lai.
Category: NT

Problem 45. Josh is trying to beat a 10 world challenge in a video game without dying. If he has a $\frac{11-n}{13-n}$ chance of beating world n without dying, what is the expected number of attempts it will take him to succeed? (Assume all probabilities are independent of each other.)

Solution. His probability of succeeding on any attempt is

$$\frac{10}{12} \cdot \frac{9}{11} \cdot \frac{8}{10} \cdots \frac{2}{4} \cdot \frac{1}{3} = \frac{2 \cdot 10!}{12!} = \frac{2}{11 \cdot 12} = \frac{1}{66}$$

Thus the expected number of attempts is the reciprocal of this, which is $\boxed{66}$. ■

Proposed by Joshua Pate.
Category: Algebra

Problem 46. William is taking a practice MATHCOUNTS sprint. He has a $\frac{1}{n+1}$ chance of getting problem n correct. (Assume all problems are independent.) What is the probability that he scores a 0? Express your answer as a common fraction.

Solution. The probability that he scores a zero is

$$\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdots\frac{30}{31}$$

which telescopes to $\boxed{\frac{1}{31}}$. ■

Proposed by Joshua Pate.
Category: Algebra

Problem 47. How many points with integer coordinates are inside or on the rectangle bounded by the lines $x = 21$, $x = 43$, $y = 17$, and $y = 26$?

Solution. Since we need to count points on the boundary, the product we seek is $(43-21+1)(26-17+1) = (23)(10) = \boxed{230}$. ■

Proposed by Jeffrey Huang.
Category: Arithmetic, Algebra, Counting, Geometry

Problem 48. A coin store consists of only pennies, nickels, dimes, and quarters. The least number of coins required to form a total value of n cents is greater than the least number of coins to form a total value of $n+1$ cents. What is the minimum possible value of n?

Solution. We require 4 pennies compared to 1 nickel, so the answer is $\boxed{4}$. ■

Proposed by Jeffrey Huang.
Category: Trivia

Problem 49. Tracy chooses a random positive integer less than or equal to 2020 and repeatedly squares it 2020 times. Compute the probability the last digit is 1 after Tracy is done squaring. Express your answer as a common fraction.

Solution. The last digit is the only information we care about, and it is uniformly distributed among the positive integers less than or equal to 2020. Notice that if Tracy starts with an even number, then squaring won't ever change the number's parity, so the last digit cannot be 1 after Tracy is done squaring. Similarly, if Tracy starts with a number divisible by five, then five will divide whatever number Tracy has after squaring, so the last digit will be 0 or 5, not 1. So to achieve 1, Tracy's last digit must start with 1, 3, 7, or 9. And for each of these, we find that after a few squares, the end on 1:

$$3,7 \longmapsto 9 \longmapsto 1 \longmapsto 1 \longmapsto \cdots \longmapsto 1.$$

After 2020 squarings, each goes to 1. The probability that Tracy began with a 1, 3, 7, or 9 as her last digit is $\frac{4}{10} = \boxed{\frac{2}{5}}$. ■

Proposed by Nir Elber.
Category: NT

Problem 50. What is smallest non-prime positive integer that is relatively prime to every prime number less than 100?

Solution. Because 1 is relatively prime to all positive integers, the desired answer is $\boxed{1}$. ■

Proposed by Jeffrey Huang.
Category: NT

Problem 51. There are 1000 lockers numbered 1 - 1000. Suppose you open all of the lockers, then close every other locker. Then, for every third locker, you close each opened locker and open each closed locker. You follow the same pattern for every fourth locker, every fifth locker, and so on up to every thousandth locker. How many lockers are touched exactly three times?

Solution. On your n-th pass, you touch only the lockers whose numbers are divisible by n. Thus the number of times you touch a locker equals the number of positive divisors its number has. Thus, a locker that is touched exactly three times has a number n with exactly three positive divisors. So $n > 1$, and it is divisible by both n and 1, and also by exactly one other thing d with $1 < d < n$. Now the number on the locker is also divisible by $\frac{n}{d}$, which is strictly between 1 and n, so we must in fact have $\frac{n}{d} = d$ and thus $n = d^2$. Moreover, d must be prime, or else it has a divisor strictly between 1 and itself which is also a divisor of n other than the three we claim exist. So n has three positive divisors if and only if it is the square of a prime p – its divisors in this case are $1, p, p^2$. Now it only remains to count how many squares of primes there are between 1 and 1000. Since $31^2 = 961 < 1000 < 1024 = 32^2$, we just need to count the primes up to 31. These are

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31$$

and there are $\boxed{11}$ of them. ■

Proposed by Joshua Pate.
Category: NT

Problem 52. The sum of the consecutive positive integers between a and b, inclusive, where $a \le b$, is 2021. What is the minimum possible value of a?

Solution. We can have 1, 43, 47, 2, 86, or 94 positive integers. The sum of the first 70 positive integers is already $70 \times 71/2 = 4970/2 = 2485 > 2021$, so we can rule out the latter two values. The best value to take is 47 positive integers, which happens when the middle, or 24^{th} term, is 43. It follows that the smallest integer in this sequence is $43 - 23 = \boxed{20}$. ■

Proposed by Jeffrey Huang.
Category: Algebra, NT

Problem 53. Queen Qua quizzes Quirky Quo on quixotic quadratics for quartz qubits. Correct quips to one of Queen Qua's quiz questions qualifies Quirky Quo a quintuple of Quo's current quartz qubits. Incorrect quips to quiz questions (queerly) qualifies quadrupling. Quo starts with 1 quartz qubit. How many correct and incorrect quips, combined, to Queen Qua's quiz questions are necessary for Quirky Quo to qualify for exactly 1600 quartz qubits?

Solution. Because $1600 = 2^4 \cdot 5^2 \cdot 2^2 = 5^2 \cdot 4^3$, Quo needs to quintuple his total 2 times and quadruple 3 times. This totals to $\boxed{5}$ total questions. ■

Proposed by Nir Elber.
Category: NT
Remark: This is a bad problem, and I am a bad person.

Problem 54. On a long staircase, Jay runs 8 steps at a time, starting from step 10, and Kay runs down 13 steps at a time, starting from step 1010. How many steps are stepped on by both Jay and Kay?

Solution. Notice that step 1010 is stepped on by both Jay and Kay from the modulo relation. Then we divide 1010 by $8 \times 13 = 104$ to see how many other steps Jay and Kay both touch. We see that $1010/104 = 9 + 74/104$, so they both step on a total of $9 + 1 = \boxed{10}$ steps. ■

Proposed by Jeffrey Huang.
Category: NT
Remark: too many 10s

Problem 55. Josh owns a giant melon farm. Every acre of farm produces 100 melons every day. If Josh's farm consists of a square plot of land 300 miles long (and a square mile consists of 640 acres) then Josh produces exactly N million melons every hour. What is N?

Solution. Josh's farm is 57 600 000 acres large, so he produces 5 760 000 000 melons every day. Hence, every hour he produces $\boxed{240}$ million melons. ■

Proposed by Pierce Lai.
Category: Algebra

Problem 56. Rich has two identical phones. Initially, one is at full battery while the other is dead (empty battery). Rich charges one while using the other until all the battery runs out. Phones deplete their entire battery at a uniform rate in 6 hours. Phones charge their entire battery in 10 hours. How many hours does it take for him to run out of battery?

Solution. He has a total of 1 whole battery. He depletes total battery at a rate of $\frac{1}{6} - \frac{1}{10} = \frac{1}{15}$, so it takes $\boxed{15}$ hours. ■

Proposed by Ethan Liu.
Category: Algebra

Problem 57. What is the least possible integer value of $\sqrt{x + \sqrt{y + 2021}}$, where x and y are positive integers?

Solution. One can see that we require $\sqrt{y + 2021}$ to be an integer. Hence, it must be at least 45. The implication is that $\sqrt{x + \sqrt{y + 2021}} \geq \sqrt{x + 45}$. The smallest positive integer greater than or equal to $\sqrt{45}$ is $\sqrt{49} = \boxed{7}$. ■

Proposed by Jeffrey Huang.
Category: Algebra

Problem 58. Emma rolls two fair 10-sided dice labeled with the positive integers 1 through 10. What is the probability that their sum is a prime number? Express your answer as a common fraction.

Solution. The prime numbers that can be a sum of two such dice are 2, 3, 5, 7, 11, 13, 17, 19. There are a total of $1 + 2 + 4 + 6 + 10 + 8 + 4 + 2 = 37$ ways to attain such a sum, so the probability is $\boxed{\dfrac{37}{400}}$. ■

Proposed by Jeffrey Huang.
Category: Combo, NT

Problem 59. Compute the number of integers no more than 1024 which can be written as the sum of exactly two (not necessarily distinct) powers of 2.

Solution. The main idea is to use binary. Let 2^a and 2^b be the powers of 2 we are interested in, with a and b nonnegative integers. Because $2^a, 2^b < 2^a + 2^b \le 1024$, we see that $a, b < 10$, and indeed

$$2^9 + 2^9 = 1024$$

is permissible, so we may let a and b range over nonnegative integers less than 10 and proceed to not care about the 1024 bound. Most of the time, we are essentially choosing two bits within 10 of the decimal point and flipping them; for example, with $a = 0$ and $b = 6$, this looks like

$$2^0 + 2^6 = \texttt{00010\ 00001}_2.$$

So provided $a \ne b$, the number of integers we generate this way is $\binom{10}{2}$.

There are an additional 10 possible integers with $a = b$ varying over nonnegative integers less than 10. (The integers with $a = b$ are distinct from the integers with $a \ne b$ because $a = b$ gives only one bit in the binary expansion, but $a \ne b$ gives two.) It follows that our final answer is $\binom{10}{2} + 10 = \boxed{55}$. ■

Proposed by Nir Elber.
Category: Combo

Problem 60. On Tuesday, the price of the IT-21 calculator increased by 20% from its price on Monday. On Wednesday, the price of the IT-21 calculator increased by 21% from its price on Tuesday. The price of the IT-21 calculator on Wednesday was what percent higher than the price of the calculator on Monday? Express your answer as a decimal to the nearest tenth.

Solution. We see that $1.21 \times 1.2 = 1.452$, so the answer is $\boxed{45.2\%}$. ■

Proposed by Jeffrey Huang.
Category: Algebra

Problem 61. A bus driver enters an empty bus and drives it to pick up 20 passengers at Orangeville, 21 passengers at Greenville, and 45 passengers at Purpleville. After picking up the passengers at the last stop, how many people are in the bus in total?

Solution. There are $20 + 21 + 45 = 86$ passengers plus 1 for the bus driver. Hence, there are a total of $\boxed{87}$ people on the bus. ■

Proposed by Jeffrey Huang.
Category: Arithmetic

Problem 62. Elsa is playing hide-and-seek in the park. This very large park is a regular hexagon with side length 10 meters, and there are lampposts at each vertex of the hexagon. Elsa can only hide if she is at least 5 meters away from the nearest lamppost. What area of the park can she hide in? (Assume that the size of Elsa does not matter.) Express your answer in simplest radical form in terms of π.

Solution. We see that the available area of the park is equal to the area of the whole park minus six circular sectors. The area of the whole park is $150\sqrt{3}\text{m}^2$, and the six circular sectors can be put together to form two circles of radius 5m, for total area 50π. Thus, the final answer is $\boxed{150\sqrt{3} - 50\pi}\,\text{m}^2$. ■

Proposed by Pierce Lai.
Category: Geo

Problem 63. A jug of juice contains roughly 5 liters of juice. A fruit fly can drink roughly 0.005 milliliters of juice. How many jugs' worth of juice can a swarm of 100 million fruit flies drink?

Solution. The fruit flies can drink 500 liters of juice together, or $\boxed{100}$ jugs' worth. ■

Proposed by Pierce Lai.
Category: Algebra

Problem 64. Two equilateral triangles of side length 3 are inscribed in a single circle. One triangle is rotated 60 degrees from the other, so the two together form a symmetrical six-pointed star. What is the area of the region contained in the circle but not in either triangle? Express your answer in simplest radical form in terms of π.

Solution. By inspection, we see that each triangle essentially trisects all three sides of the other triangle, so the region of each triangle that does not overlap with the other is just 3 equilateral triangles with side length 1. The total area contained in the two triangles is thus $12 * \sqrt{3}/4 = 3\sqrt{3}$. The radius of the circle is equal to the height of two small triangles, or $\sqrt{3}$, so the area of the circle is 3π, so the final answer is $\boxed{3\pi - 3\sqrt{3}}$.
■

Proposed by Pierce Lai.
Category: Geo

Problem 65. There are fifty teams competing at the Hardly Meaningful Math Olympiad (HMMO). Matthew is a coach of one of these teams, but for some reason he is not permitted to watch the spectator events or see the rankings. Because of this, he has no idea how his team is going to rank against the other teams, and assumes all rankings from 1st to 50th are equally likely. After the contest, one of Matthew's students cryptically tells him that their ranking was a perfect square. Given that information, what is the probability that the team placed in the top ten? Express your answer as a common fraction.

Solution. There are three perfect squares from 1 to 10, and seven (up through $7^2 = 49$) from 1 to 50. Thus the probability is $\boxed{\frac{3}{7}}$. ■

Proposed by Matthew Kroesche.
Category: Combo

Problem 66. What is the value of $99^3 + 2 \times 99^2 + 2 \times 99 + 1$?

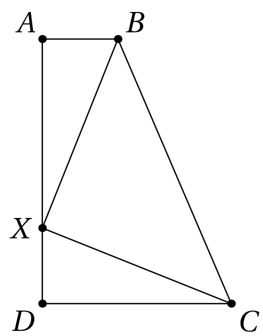
Solution. Set $x = 99$. Then $x^3 + 2x^2 + 2x + 1 = (x+1)(x^2 + x + 1) = 100 \times 9901 = \boxed{990,100}$. ■

Proposed by Jeffrey Huang.
Category: Algebra

Championship Problems

Problem 67. Trapezoid $ABCD$ satisfies $\angle BAD = \angle ADC = 90^\circ$ and $AB = 2, CD = 5$, and $DA = 7$. A point X is chosen on AD such that $\angle BXC = 90^\circ$. Compute the largest possible value of AX.

Solution. Note the diagram.



We need $\triangle BXC$ to be a right triangle, which is equivalent to

$$BX^2 + CX^2 = BC^2.$$

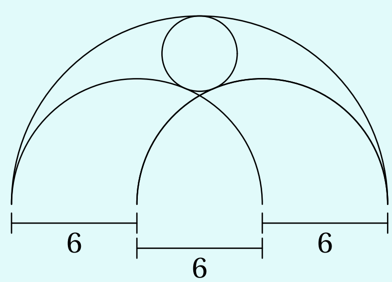
Fix $x = AX$. To be concise, let $a = AB = 2, b = AD = 7$, and $c = CD = 5$. Using the other right triangles in the figure, we see

$$a^2 + x^2 + (b - x)^2 + c^2 = b^2 + (c - a)^2.$$

After cancellation, this looks like $2x^2 - 2bx + 2ac = 0$. Plugging in, this tells us $x^2 - 7x + 10 = (x - 2)(x - 5) = 0$. The largest possible value is $\boxed{5}$. ■

Proposed by Nir Elber.
Category: Geometry

Problem 68. Three points lie along a line such that the distance between each pair of consecutive points is 6. Three semicircular arcs are constructed between all non-adjacent points, as in the picture below. A smaller circle is externally tangent to the two arcs of radius 6 and internally tangent to the arc of radius 9. What is its radius? Express your answer as a common fraction.



Solution. Look at the right triangle from the three points A, the center of the big circle, B, the center of one of the median sized circles, and C, the center of the smallest circle. Pick r to be the radius of the small circle. Because $\circ B$

is tangent to $\circ C$, $BC = 6 + r$. Similarly, $AC = 9 - r$ and $AB = 9 - 6$. Thus,

$$\begin{aligned}(9-r)^2 + 3^2 &= (6+r)^2 \\ 0 &= 36 + 12r - 81 + 18r - 9 \\ &= 30r - 54 \\ r &= \boxed{\frac{9}{5}},\end{aligned}$$

■

Proposed by Eli Meyers.
Category: Geometry

Problem 69. Each of the faces of a $5 \times 5 \times 5$ cube is numbered according to the scheme below and then split into unit cubes.

1	2	3	2	1
2	3	4	3	2
3	4	5	4	3
2	3	4	3	2
1	2	3	2	1

A unit cube is chosen arbitrarily. What is the probability that the sum of the numbers on the faces of the cube is at least 4? Express your answer as a common fraction.

Solution. There are $125 - 27 = 98$ cubes with at least one face numbered. There are 8 corner pieces, each of which has a number 1 three times, not enough. The edge pieces are fine, but there are 4 unit cubes per face with only one face numbered, and numbered with 3. Hence, there are $98 - 8 - 24 = 66$ valid cubes. In conclusion, the answer is $\boxed{\frac{66}{125}}$. ■

Proposed by Jeffrey Huang.
Category: Combo, Geo
Remark: Use the tabular environment or something to clean this up

Problem 70. The second time after 12:00 PM that the smaller angle between the hour hand and the minute hand on a clock is 85 degrees, occurs N minutes after 12:00 PM. Find N.

Solution. We see that every minute, the minute hand moves by 6 degrees while the hour hand moves by .5 degrees, so the overall change is 5.5 degrees a minute. At 12:00 the angle between the two hands is 0, so the second time 85 degrees occurs is when the total change is $360 - 85 = 275$ degrees, and so the time passed is $275/5.5 = \boxed{50}$ minutes. ■

Proposed by Pierce Lai.
Category: Algebra

Problem 71. In a math bowl, team A and team B each have 4 members. Each member of team A will individually face off against a different member of team B, and a team wins the match if they have strictly more wins than the opposing team. Each of the 8 competing members can beat a unique set of 2 members of the opposing team. How many ways are there to pair the students so that team A wins the match?

Solution. Because this setting is symmetric with respect to A and B, we count the number of ways to tie. After some inspection, we see that WLOG, we can set Team A to beat Team B if in modulo 4, $a-b$ equals 0 or 1. A Adding up the values, we see that the sum of such $a-b$ values equals 0, 4, 8, or 12. The only unordered quadruples that work are $(2,2,0,0)$ and $(3,3,1,1)$. In the first case, we choose either 0 and 2 be the identity map or 1 and 3 to be the identity map. For the other case, we are flipping either about $x+y=3$ or $x+y=1$. Hence, there are a total of 4 possible ties. The number of ways A can win over B is $\frac{4!-4}{2} = \boxed{10}$. ■

Proposed by Jeffrey Huang.
Category: Combo
Remark: too many 10s

Problem 72. Let a,b,c,d,e be distinct digits from 1-9. If $\overline{abc}_{\overline{de}}$ (where $\overline{abc}$ is a 3 digit number in the 2 digit base $\overline{de}$, where $\overline{de}$ is in base 10) is equal to 2021 base 10, what is the smallest value of the five digit number $\overline{abcde}$?

Solution. For $abcde$ to be as small as possible, we want $a=1$. Then we have $de^2+b*de+c=2021$. We then want to minimize b, which means maximizing de. $45^2=2025>2021$, so we then test $de=43$. We have the $2021=43*47=43^2+4*43$, so 43 doesn't work (since b must be distinct from d). Then, we test $de=42$, which works as $42^2+42*6+5=2021$. Hence, the answer is $\boxed{16542}$.

■

Proposed by Pierce Lai.
Category: NT

Problem 73. There are ten pieces of mutton, but two of them contain iocane powder, one of the more deadly poisons known to man. A chef suspects this, and tests five different pieces of mutton at random (without consuming them); upon finding that a piece is poisoned, the chef throws it out. What is the probability that no more poisoned mutton remains? Express your answer as a common fraction.

Solution. Label the pieces of mutton $1-10$, with 1 and 2 being poisoned. The number of ways such that the pieces 1 and 2 are picked is $\binom{8}{3}$, and the total number of ways to select five pieces of mutton is $\binom{10}{5}$. Thus, the answer is $\frac{\binom{8}{3}}{\binom{10}{5}} = \frac{56}{252} = \boxed{\frac{2}{9}}$. ■

Proposed by Josiah Kiok.
Category: Combo

Problem 74. Let ABC be an isosceles triangle such that $A=(0,0)$, $B=(3,1)$, C is a point with integer coordinates, and the area of triangle ABC is an integer. What is the square of the minimum possible length of AC?

Solution. If AB is the base of the triangle, then the minimum possible area comes from $C=(1,2)$, which has area $\frac{5}{2}$, not an integer. All other lattice points will give an area of $\frac{5}{2}+5k$, where k is an integer. This is also never an integer. The only other possibility is $AC=AB=\sqrt{10}$, attained by setting $C=(-1,3)$. Hence, the answer is $\sqrt{10}\cdot\sqrt{10}=\boxed{10}$. ■

Proposed by Jeffrey Huang.
Category: Geometry
Remark: too many 10s

Problem 75. Suppose Pierce can write a problem every 5 minutes, Josh can write a problem every 4.5 minutes, Matthew can write a problem every 4 minutes, and Nir can write a problem every 3.5 minutes. If they all start writing problems at the exact same time, how many problems will the four have written by the first instance when they all finish writing a problem at the same time?

Solution. We essentially want 1/2 of the lcm of 10, 9, 8, and 7, which we very quickly see is 2520, so 1/2 of that is 1260. Hence, in this time, Pierce will have written 252 problems, Josh will have written 280 problems, Matthew will have written 315 problems, and Nir will have written 360 problems, so in total they will have written $252+280+315+360=\boxed{1207}$ problems. ■

Proposed by Pierce Lai.
Category: NT

Problem 76. Diana writes down all divisors of 81 on a sheet of red paper. Then, for each number on the red paper, she writes down all its divisors on the blue paper, including duplicates. The product of all numbers on the blue paper can be written as 3^k. Compute k.

Solution 1. Note $81=3^4$. Now tabulate our divisors. We place blue divisors directly below the red paper.

red paper	1	3	3^2	3^3	3^4
blue paper	1	1	1	1	1
		3	3	3	3
			3^2	3^2	3^2
				3^3	3^3
					3^4

Taking products by columns, the total product is $1\cdot 3\cdot 3^3\cdot 3^6\cdot 3^{10}=3^{20}$. So the answer is $\boxed{20}$. ■

Solution 2. We overkill. Note that $81=3^4$, so all numbers Diana writes down will look like 3^r for some nonnegative integer r. In particular, the red paper features $r=0$ up to $r=4$. It follows that the product of the numbers on the blue paper is

$$\prod_{r=0}^{4}\left(\prod_{b=0}^{r}3^b\right).$$

Moving everything to the exponent, this product is

$$3^{\sum_{r=0}^{4}\sum_{b=0}^{r}b}=3^k.$$

We want the exponent of 3 here. The summation is the fourth tetrahedral number, so

$$k=\frac{4\cdot 5\cdot 6}{6}=\boxed{20}$$

using the tetrahedral number formula. So we are done here. ■

Proposed by Nir Elber.
Category: NT

Problem 77. The height of an isosceles trapezoid with integer side lengths is an integer. What is the minimum possible perimeter of the trapezoid, in units?

Solution. The height must be part of a Pythagorean triangle with integer leg lengths. We deduce that the leg lengths are 5 units. We can also set the shorter base equal to 1 unit. The longer leg length will be $1+2g$ units, where g is one of the legs of the Pythagorean triangle. The minimum possible value of g is 3, giving us our minimum perimeter of $1+5+5+7=\boxed{18}$ units. ■

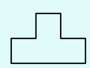
Proposed by Jeffrey Huang.
Category: Geometry

Problem 78. How many ordered triples of positive integers (x, y, z) satisfy $x + xy + xyz = 18$?

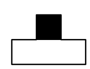
Solution. Factor the left side as $x(1 + y + yz)$. We know that $1 + y + yz \geq 3$, so $x = 1, 2, 3, 6$. If $x = 6$, then $y + yz = 2$ has only 1 solution. If $x = 3$, then $y(1 + z) = 5$ has only 1 solution. If $x = 2$, then $y(1 + z) = 8$ has 3 solutions. Finally, if $x = 1$, then $y(1 + z) = 17$ has only 1 solution. There are a total of $\boxed{6}$ solutions. ■

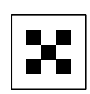
Proposed by Jeffrey Huang.
Category: Algebra, NT

Problem 79. Justin has a 5×5 checkerboard grid, so every square is colored red or black in such a way that adjacent squares are different colors and the center square is red. Justin wants to place a tetromino in the shape of a T (as shown below) completely on the grid, covering up some squares of the board in such a way that exactly 10 red squares are showing, and the edges of the tetromino line up with the edges of adjacent tiles on the board. How many ways are there for Justin to place it following these criteria?



Solution. There are 5 spaces in the middle to put the filled square below in which it can be rotated 4 ways (the middle cross).





There are also 4 spaces on the side where you can put that piece which each allow for 1 orientation of the T piece. Thus, the answer is $5 * 4 + 4 = \boxed{24}$. ■

Proposed by Eli Meyers.
Category: Combo

Problem 80. Matthew, Justin, and Heffrey are discussing their ages. In 6 years, Matthew will be twice as old as Justin will be. In 3 years, Justin will be 2/3 as old as Heffrey will be. 18 years ago, Matthew was twice as old as Heffrey. What is the sum of their ages 1 year from now?

Solution. Letting M, J, and H be the current ages of Matthew, Justin, and Heffrey respectively, we get the three equations $M = 2J + 6$, $J = 2/3H - 1$, and $M = 2H - 18$. We get that $4/3H + 4 = 2H - 18$, so $H = 33$. Thus, $J = 21$ and $M = 48$. Hence, the sum of their ages 1 year from now is $33 + 21 + 48 + 3 = \boxed{105}$. ■

Proposed by Pierce Lai.
Category: Algebra

Problem 81. Compute the sum of all n for which $n!$ is divisible by 5^3 but not 5^4.

Solution. Note that $(n+1)!$ will have more powers of 5 than $n!$ if and only if $n+1$ is divisible by 5 because $(n+1)! = (n+1)\cdot n!$. So further, to count the number of powers of 5 in $n!$, we need only consider the set of integers divisible by 5 below n and the number of 5s dividing them. This sequence of number is

$$5, 10, 15, 25, \ldots.$$

To be explicit, we just do the casework.

(a) For $5 \le n < 10$, we have that $n!$ only has to worry about 5, so we get one power of 5.

(b) For $10 \le n < 15$, we have that $n!$ has to worry about 10 also, so we get two powers of 5.

(c) For $15 \le n < 20$, we have that $n!$ has to worry about 15 also, so we get three powers of 5.

(d) For $n \ge 20$, $n!$ is divisible by $5 \cdot 10 \cdot 15 \cdot 20$ and so is divisible by $5 \cdot 5 \cdot 5 \cdot 5 = 5^4$.

It follows that the n we want are $15+16+17+18+19 = 5\cdot 17 = \boxed{85}$. So we are done here. ■

Proposed by Nir Elber.
Category: NT
Remark: I think this problem is too hard, but I'm comparing with 2020 State Sprint/8.

Tiebreaker Round

Problem 1. Geoff and Jeff each independently write down a list of positive integers from 1 to 6 inclusive, such that each person's list is sorted from smallest to greatest, no number appears more than once on a person's list, and neither list is empty. (It is possible for a number to appear on both lists, on just one, or on neither.) Then, Heff makes another list, consisting of all products of the form gj, where g is a number from Geoff's list and j is a number from Jeff's list. (Numbers can appear on Heff's list more than once; thus the length of Heff's list is exactly the length of Geoff's list times the length of Jeff's.) In how many ways can Geoff and Jeff pick their lists such that exactly half of the numbers on Heff's list are odd?

Solution. Let S denote the set of Geoff's numbers, and T the set of Jeff's numbers. Pick S_o and T_o to be the sets consisting of the odd numbers from S and T respectively. The total number of odd products will be $|S_o| \cdot |T_o|$, because the only way to get an odd product is an odd number times an odd number. Obviously, the total number of products is going to be $|S| \cdot |T| = |S_o| \cdot |T_o| + (|S| - |S_o|) \cdot |T_o| + |S_o| \cdot (|T| - |T_o|) + (|S| - |S_o|) \cdot (|T| - |T_o|)$. Pick a, b, c, and d to be $|S_o|$, $|T_o|$, $|S| - |S_o|$, and $|T| - |T_o|$ respectively. This means

$$ab = cd + ad + bc.$$

Let n_s be the number of odd numbers in set S and n_t the number of odd numbers in set T. Then, the proportion of odd products is $\frac{n_s}{|S|}\frac{n_t}{|T|}$, which is equal to $\frac{1}{2}$. We have that $n_s, n_t \leq 3$, $|S| \leq n_s + 3$, and $|T| \leq n_t + 3$. Thus, we can write out all possible fractions (using the fact that each fraction has to be at least $\frac{1}{2}$, or else their product won't be $\frac{1}{2}$):

$$\frac{1}{1}, \frac{1}{2}, \frac{2}{2}, \frac{2}{3}, \frac{2}{4}, \frac{3}{3}, \frac{3}{4}, \frac{3}{5}, \frac{3}{6}$$

Now of these, $\frac{1}{1}, \frac{2}{2}, \frac{3}{3}$ are equal to 1, and $\frac{1}{2}, \frac{2}{4}, \frac{3}{6}$ are equal to $\frac{1}{2}$. The remaining ones are $\frac{2}{3}$, $\frac{3}{4}$, and $\frac{3}{5}$. Hence, the fractions corresponding with S and T can either equal 1 and $\frac{1}{2}$, or $\frac{2}{3}$ and $\frac{3}{4}$.

Finally, we need to count all the possibilities. There are 3 ways to choose a set with $\frac{1}{1}$, 3 for $\frac{2}{2}$, 1 for $\frac{3}{3}$, 9 for $\frac{1}{2}$, 9 for $\frac{2}{4}$, and 1 for $\frac{3}{6}$. Hence, the number of possibilities for 1 and $\frac{1}{2}$ is $2(3+3+1)(9+9+1) = 266$. There are 9 for $\frac{2}{3}$ and 3 for $\frac{3}{4}$, so that gives $2(9)(3) = 54$. Thus, the solution is $266 + 54 = \boxed{320}$. ■

Proposed by Eli Meyers.
Category: Combo
Remark: solved by Pierce Lai

Problem 2. What is the sum of the positive integer divisors of 20210?

Solution. Since $2021 = 2025 - 4 = 45^2 - 2^2 = 43 * 47$, the sum of the divisors is $(2+1)(5+1)(43+1)(47+1) = 3 * 6 * 44 * 48 = 3 * 6 * 4 * 48 * 11 = 8 * 432 * 11 = 3456 * 11 = \boxed{38016}$. ■

Proposed by Joshua Pate.
Category: NT
Remark: I think this is harder than we think it is. Namely, I don't think contestants should be required to know this formula at the easy level, and there are a few too many divisors to make the arithmetic easily manageable. – Nir

Problem 3. Call a positive integer *messy* if it has at least five factors. What is the third lowest messy number?

Solution. We can solve this by inspection: the three lowest messy numbers are 12, 16, and 18. Thus, the answer is $\boxed{18}$. ■

Proposed by Josiah Kiok.
Category: NT

Challenge Round

Problem A1. Frederic was born some number of years ago. What is the least positive integer N such that, on Frederic's birthday when he turns N years old, it is guaranteed that he has had a birthday (including the day he was born) fall on every day of the week at least once?

Solution. Our first instinct would be to say the answer to this question is **10**. The reason being that a non-leap year has 365 days, which is 1 more than a multiple of 7, and a leap year has 366 days, which is 2 more than a multiple of 7. So WLOG suppose Frederic was born on January 1, and suppose WLOG that the last day of the week to fall on his birthday is a Sunday. Then he has to "jump" over Sunday at some point, that is, his birthday should be on a Saturday one (leap) year, and on a Monday the following year. (For convenience here and throughout, denote the days of the week as Sunday=0 through Saturday=6.) Then the days that Frederic's birthday falls on before and after that leap year are (with the leap year highlighted in bold)

$$\cdots, 3, 5, 6, 0, 1, 3, 4, 5, \mathbf{6}, 1, 2, 3, 4, 6, 0, 1, 2, 4, \cdots$$

So the farthest back we can go without hitting the day we skipped over is four years, and the farthest forward we can go is five years. Thus, by the time Frederic turns ten, he will be sure to have had a birthday fall on Sunday.

But this is wrong. The reason (well, one of them) is that years that are divisible by 100 but not by 400 are not actually leap years. Suppose Frederic was born on January 2, 1899, which was a Monday. Then his birthday in 1900 will be a Tuesday, his birthday in 1901 will be a *Wednesday* (because 1900 is not a leap year), and his first twelve birthdays will go

$$1, 2, 3, 4, 5, 6, 1, 2, 3, 4, 6, 0$$

Thus he turns eleven on January 2, 1910, and that is the first time his birthday falls on a Sunday. So we may be tempted to say the answer is **11**.

But *this* is wrong. The reason (and it's a really evil one) is that Frederic may, in fact, have been born on February 29th of a leap year. (Note that the problem is carefully worded to not rule out this possibility.) Now there are $365 \cdot 4 + 1 = 1461$ days between two of Frederic's birthdays that are in two consecutive leap years, and this number is 5 more than a multiple of 7. So for Frederic to even have had seven birthdays, he will have to be at least 24 years old. Under normal circumstances, every one of those seven birthdays will fall on a different day of the week (because 5 is relatively prime to 7) and so we may be tempted to say that the answer is **24**.

But this is also wrong, as you may have guessed by now. Remember that some years that are divisible by four are not actually leap years. If Frederic was born on February 29th of a leap year and lives through, say, the year 1900, 2921 days will pass between two of his consecutive birthdays, and this number is 2 more than a multiple of 7. Thus Frederic's birthday in 1904 will be on the same day that his birthday in 1892 was on. So his date counter goes backwards. To take advantage of this phenomenon, for concreteness suppose Frederic was born on February 29, 1884 (which happens to be a Friday, although the actual day of the week is not important.) Then the dates of Frederic's birthdays will be

Year	Birthday
1884	Friday
1888	Wednesday
1892	Monday
1896	Saturday
1904	Monday
1908	Saturday
1912	Thursday
1916	Tuesday
1920	Sunday

So Frederic's birthday in 1920, which is his ninth actual birthday (including the day he was born), is the first time his birthday falls on a Sunday, and he turns 36 on that day. So it seems the answer should be $\boxed{36}$, and indeed, as far as I know, there is no way to improve this any further. (At least not assuming the Gregorian calendar.) ■

Proposed by Matthew Kroesche.
Category: Algebra, NT
Remark: This is absolutely the most evil thing I've ever written. Also bonus points if you recognize where the character's name came from.

Problem A2. Given that the sum of the reciprocals of all the positive perfect squares is $\frac{\pi^2}{6}$, find the sum of the reciprocals of all the odd positive perfect squares.

Solution. Let E be the sum of the reciprocals of the even squares, and O be the sum of the reciprocals of the odd squares. We know $E+O$ is the sum of the reciprocals of all the squares, which is given to be $\frac{\pi^2}{6}$. We also know

$$E = \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \cdots = \frac{1}{4}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right]$$

so E is just $\frac{1}{4} \cdot \frac{\pi^2}{6} = \frac{\pi^2}{24}$. Thus

$$O = \frac{\pi^2}{6} - \frac{\pi^2}{24} = \boxed{\frac{\pi^2}{8}}$$

■

Proposed by Joshua Pate.
Category: Algebra/NT

Problem A3. Four pirates had a pile of gold doubloons. During the night, one man woke up and decided to take his share of the booty. He divided them into four piles and took one pile for himself. One doubloon was left over so he threw it into the sea. Soon a second man woke up and did the same thing. After dividing the booty into four piles, one doubloon was left over which he threw into the sea. The third and fourth man followed exactly the same procedure. The next morning, after they all woke up, they divided the remaining doubloons into four equal shares. One remained, which they threw into the sea. What is the least number of doubloons there could have been in the original pile?

Solution. For brevity, let $f(N) = \frac{3}{4}(N-1)$, which is the operation applied by one of the pirates when dividing up N doubloons. The assertion is that all of

$$N,\ f(N),\ f^2(N),\ f^3(N),\ f^4(N),\ f^5(N)$$

are (positive) integers. Because all of our denominators are powers of 4, we could expand this into a system of modular equivalences in N, using $\pmod 4$, $\pmod{4^2}$, $\pmod{4^3}$, $\pmod{4^4}$, and $\pmod{4^5}$. In particular, because this is just a modular system, we see that N works if and only if the entire congruence class $N \pmod{1024}$ works. Further, this congruence class will be unique with this property because $f^5(N)$ really does have $\pmod{1024}$.

Anyways, the point is that it suffices to find a single N which satisfies and then reduce $\pmod{1024}$. Well,

$$f(-3) = \frac{3}{4}(-3-1) = -3,$$

so $f^\bullet(-3)$ are all integers. It follows $-3 \equiv \boxed{1021}$ is our smallest positive answer satisfying. One can check with bounding (or manually) that 1021 does indeed make the $f^\bullet$ equal to positive integers. ■

Proposed by Joshua Pate.
Category: Algebra

Problem A4. Matthew flips a fair coin until it lands heads heads heads, or it lands tails heads tails. What is the probability that he stops because he flips heads heads heads?

Solution. Let p_{HT} be the probability that Matthew stops because he flips heads heads heads, given that his last two flips were heads and tails, in that order. Define p_{HH}, p_{TT}, p_{TH} similarly. Now if his last two flips were both heads, there's a $\frac{1}{2}$ chance he flips heads again and wins; otherwise he flips tails and goes to p_{HT}. Thus

$$p_{HH} = \frac{1 + p_{HT}}{2}$$

If his last two flips were tails heads, there's a $\frac{1}{2}$ chance he loses by flipping tails; otherwise he flips heads and goes to p_{HH}. Thus

$$p_{TH} = \frac{p_{HH}}{2}$$

The other two cases follow similarly:

$$p_{HT} = p_{TT} = \frac{p_{TH} + p_{TT}}{2}$$

Putting everything in terms of $p = p_{TT}$, we see that

$$p = \frac{p}{2} + \frac{p_{TH}}{2} = \frac{p}{2} + \frac{p_{HH}}{4} = \frac{p}{2} + \frac{1+p}{8} = \frac{5p+1}{8}$$

Thus $p = \frac{1}{3}$. So

$$p_{HT} = p_{TT} = p_{TH} = \frac{1}{3}$$

$$p_{HH} = \frac{2}{3}$$

Since Matthew's first two coin flips are equally likely to be any one of these four, the probability that he wins is just the average

$$\frac{1}{4}\left(\frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{2}{3}\right) = \boxed{\frac{5}{12}}$$

■

Proposed by Joshua Pate.
Category: Algebra

Problem A5. In the addition problem

$$\begin{array}{r} GALILEO \\ GALILEO \\ GALILEO \\ GALILEO \\ GALILEO \\ +\quad FIGARO \\ \hline 20222202 \end{array}$$

each letter represents a different nonzero digit. Compute the seven-digit number *GALILEO*.

Solution. Let a, b, c, d, e, f denote the values that are carried into each of the leftmost six columns. Note that each of a, b, c, d, e, f is at least 1 and at most 5. (None of them can be zero, or else that would imply that the sum of the previous column came to at most 2, which is clearly impossible since all the digits are nonzero.).

Looking at the first column, $5G + a = 20$. Thus it must be that $a = 5$ and $G = 3$.

Now looking at the fourth column, we see that $5I + G + d$ ends with 2. Thus, since $G = 3$, $5I + d$ ends with 9. Thus it must be that $d = 4$ and I is odd.

Next we look at the third column, which tells us that $5L + I + c$ ends with 2. Now if L is even, $I + c$ ends with 2, and if L is odd, $I + c$ ends with 7. Thus, since I is odd, if L is even, it must be that $I = 1$ and $c = 1$. And if L is odd, it must be that $I = 5$ and $c = 2$. (Recall that $I \neq 3$.) But if $I = 5$, then $5I + G + d = 32$, so $c = 3$, which is a contradiction. So it must be that $I = 1$, $c = 1$, and L is even.

Next we turn our attention to the fifth column, which tells us that $5L + A + e = 42$ (because $d = 4$). Now L is even, so it's one of $2, 4, 6, 8$. It can't be 2 or 4 since $A + e$ can be at most 14. And if $L = 8$, then $A + e = 2$, which implies that they're both 1. But this is impossible since $I = 1$. So it must be that $L = 6$, and $A + e = 12$.

Since $L = 6$, we can look at the third column again to see that $5L + I + c = 32$. So $b = 3$.

Now looking at the second column, $5A + F + b = 52$ (because $a = 5$) and so $5A + F = 49$. Thus it must be that either $A = 9, F = 4, e = 3$; or $A = 8, F = 9, e = 4$ (combining this with the condition on A from the fifth column).

Now we look at the sixth column, which tells us $5E + R + f = 10e$. If $e = 4$, then we have $5E + R + f = 40$. Then it must be that $E = 7$ (because 6 is taken and 5 is too small) and $R + f = 5$. If instead $e = 3$, then we have $5E + R + f = 30$. Then it must be that $E = 5$ (because 4 and 3 are taken – if $e = 3$ then $F = 4$) and $R + f = 5$. Either way, $R + f = 5$. So our two big cases remain

$$A = 9, E = 5, F = 4, e = 3$$

$$A = 8, E = 7, F = 9, e = 4$$

Now $R + f = 5$ implies that R is one of $1, 2, 3, 4$ – but 1 and 3 are taken. So either $R = 4, f = 1$ or $R = 2, f = 3$.

Next we look at the seventh column, which tells us $6O = 10f + 2$. Thus either $O = 2$ and $f = 1$, or $O = 7$ and $f = 4$. Above, we showed that either $f = 1$ or $f = 3$. Thus we must have $O = 2, f = 1, R = 4$. But now we cannot also have $F = 4$. So out of our two cases, the second one must be the winner, and so we have

$$A = 8, E = 7, F = 9, O = 2, R = 4, e = 4, f = 1$$

and $GALILEO = \boxed{3861672}$. The completed sum looks like this:

$$\begin{array}{r} {\scriptstyle 5\,3\,1\,4\,4\,1\ \ } \\ 3861672 \\ 3861672 \\ 3861672 \\ 3861672 \\ 3861672 \\ +\ \ 913842 \\ \hline 20222202 \end{array}$$

■

Proposed by Matthew Kroesche.
Category: Algebra, NT

Problem C1. How many ways are there to pair up all of the vertices of a regular octagon such that, if you draw straight lines connecting the vertices in each pair, no two lines intersect?

Solution. We start with easier problems. For a square, there are 2 ways (pick opposite edges as pairs). For a hexagon, pick one vertex to start with. If you connect the vertex to either of the 2 adjacent vertices, you are left with a quadrilateral, so this gives $2 * 2 = 4$ ways. You can also go straight across, which splits the remaining 4 vertices into sets of 2, so this gives 1 way. (Note that pairing the vertex with one of the other two vertices results in groups of 1 and 3 vertices, which cannot be paired up.) Hence, for a hexagon, there are 5 ways total.

Finally, we go to the octagon. We can pick any point to start with. If we pair up that point with an adjacent vertex, we are left with a hexagon, which we know has 5 ways. (Although this hexagon is not regular, it still has the same number of pairings as a regular hexagon.) This gives us $2 * 5 = 10$ ways. If we pair up the point with a vertex that is 3 away, the line splits the remaining 6 vertices into groups of 2 and 4. There is one way to pair up a group of 2 and 2 ways to pair up a group of 4, so this gives us $2 * 2 = 4$ ways. Hence, there are $10 + 4 = \boxed{14}$ ways to pair up the vertices of an octagon. ■

Remark: There aren't that many ways to pair up the vertices, so it would be fairly easy to just draw out all of the possibilities. (Also, I think these are just the Catalan numbers.)

Proposed by Pierce Lai.
Category: Combo

Problem C2. Baba and Keke each independently and randomly pick a positive two-digit integer (from 10 to 99 inclusive). Find the probability that Baba's integer divides Keke's integer.

Solution. Break into cases based on the ratio of the two integers. If Keke's integer k is n times Baba's integer b, then b can be at most $\frac{99}{n}$. So the number of choices for b is $\lfloor \frac{99}{n} \rfloor - 10 + 1 = \lfloor \frac{99}{n} \rfloor - 9$. Summing this from $n = 1$ to 9 gives

$$90 + 40 + 24 + 15 + 10 + 7 + 5 + 3 + 2 = 196.$$

This is as high as the ratio can be; if one integer is at least ten times another, they cannot both have two digits. So there are 196 ways to choose b and k such that b divides k. Since the total number of possible choices (all equally likely) for b and k is $90^2 = 8100$, the probability is $\frac{196}{8100} = \boxed{\frac{49}{2025}}$. ■

Proposed by Matthew Kroesche.
Category: Combo, NT

Problem C3. Eight distinguishable people sit in a circle and all spin a spinner with an equal chance of landing on red, blue, or yellow. In how many ways can they spin such that no two adjacent people spin the same color?

Solution. To simplify the problem, we label each of the people with the letters A-H clockwise, then do casework based off of the colors of A and E. Case 1: A and E have the same color (there are 3 ways to accomplish this): We can simply do casework on the number of ways B/C/D and F/G/H could have spun, which would have the same result. If B and D are the same color, that gives two possibilities for C, and if they have different colors, there would be one possibility for C. This gives a total of $2 * 2 + 2 * 1 = 6$ ways for B/C/D to spin. Since we have to count for the colors of A and E as well, along with the symmetrical case F/G/H, we have $3 * 6 * 6 = 108$ as our count for this case. Case 2: A and E have a different color (there are 6 ways to accomplish this): We do casework again. If B and D are the same color, that gives two possibilities for C, and if they have different colors, there would be one possibility for C. This gives a total of $1 * 2 + 3 * 1 = 5$ ways for B/C/D to spin. With the same structure as the first case, we have $6 * 5 * 5 = 150$ as our count for this case. This answer is thus $\boxed{258}$. ■

Proposed by Josiah Kiok.
Category: Combo

Problem C4. Consider an arrow on a number line pointing in the positive x direction with length 1. Every second, it either moves to the right as many times as its length, or it multiplies its length by $\frac{3}{5}$ with equal probability. After a long time, the expected position of the base of the arrow is x. Compute x.

Solution. Similar to last year's Target 8, we consider insertions at the beginning instead of at the end, there being a $\frac{1}{2}$ chance of shifting the whole thing forward and there being a $\frac{1}{2}$ chance of the thing being multiplied by $\frac{3}{5}$. Let E_n be the expected location of the arrow. $E_{n+1} = \frac{1}{2}(E_n + 1) + \frac{1}{2}(\frac{3}{5}E_n) = \frac{4}{5}E_n + \frac{1}{2}$. Thus, $E_n = \sum_{k=0}^{n} \frac{4}{5}^k \frac{1}{2}$. As n approaches infinity, E_n approaches $\boxed{\frac{5}{2}}$. ■

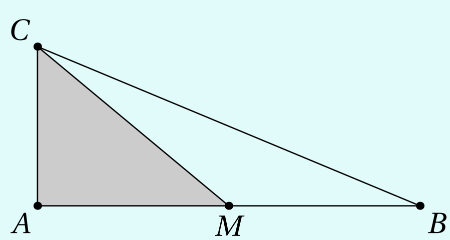
Proposed by Ethan Liu.
Category: Combo, Algebra
Remark: Did I make arrow problem 2? Yes. Is it easier? Yes. Do I regret it? No.

Problem C5. Find the number of distinct words that use the letters in "OWOUWUUWO" such that no word has more of one letter than in "OWOUWUUWO". For example, "OWOOU" and "OW" are legal while "UUWUU" is not. Words cannot be empty.

Solution. First off, an apology to anyone who has to solve this. The answer is $\sum_{i=0}^{3}\sum_{j=0}^{3}\sum_{k=0}^{3} \frac{(i+j+k)!}{i!j!k!} - 1$.
We take advantage of symmetry, namely that the number of words with 1O, 2W, and 3U is the same as the number of words with 1W,2U,3O. Thus, we have:
$0,1,2 \to 6 \cdot \frac{3!}{2!1!0!} = 18$
$0,1,3 \to 6 \cdot \frac{4!}{3!1!0!} = 24$
$0,2,3 \to 6 \cdot \frac{5!}{3!2!0!} = 60$
$1,2,3 \to 6 \cdot \frac{6!}{3!2!1!} = 360$
$0,0,1/0,0,2/0,0,3 \to 9$
$1,1,0/1,1,2/1,1,3 \to 3 \cdot (2+12+20)$
$2,2,0/2,2,1/2,2,3 \to 3 \cdot (6+30+210)$
$3,3,0/3,3,1/3,3,2 \to 3 \cdot (20+140+560)$
$1,1,1/2,2,2/3,3,3 \to 3+90+1680$
All of that sums to $\boxed{4894}$. ■

Proposed by Ethan Liu.
Category: Combo

Problem G1. Triangle ABC has $AB = 12$, $AC = 5$ and $BC = 13$, and M is the midpoint of side AB. Compute the area of triangle AMC.



Solution. Triangle ABC is a right triangle with right angle at A, since $AB^2 + AC^2 = BC^2$. Thus, triangle AMC is a right triangle with $AM = 6$ and $AC = 5$, so its area is $\frac{6\cdot5}{2} = \boxed{15}$. ■

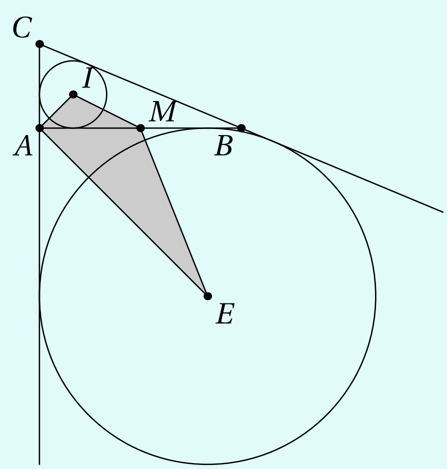
Proposed by Matthew Kroesche.
Category: Geometry

Problem G2. Let n be a positive integer at least 3. What is the probability that it is possible to construct a line segment of length a from the center C of a regular n-gon to a point P on the perimeter such that $a^2\pi$ is the area of the n-gon?

Solution. Construct the circumcircle C_1 and the incircle C_2, both centered at C. Note that the area of the incircle is $a^2\pi$ when P is picked to be the midpoint of any side. Similarly, the area of the circumcircle is $a^2\pi$ when P is picked to be any vertex. Consider the continuous function $f(\theta)$ to be $a^2\pi$ when P is the point on the perimeter with $m\angle PCV = \theta$ for some vertex V of the n-gon. $f(0)$ is just the area of the circumcircle, and $f(\pi/n)$ is just the area of the incircle. Note that because the incircle is completely contained in the n-gon is completely contained in the circumcircle, the area of the incircle is less than the area of the n-gon is less than the area of the circumcircle. From the intermediate value theorem, there must be some θ such that $f(\theta)$ is the area of the n-gon. Thus, for any value of n we can pick a θ that works. The answer is $\boxed{1}$. ■

Proposed by Eli Meyers.
Category: Geometry

Problem G3. Triangle ABC has $AB = 12$, $AC = 5$ and $BC = 13$, and M is the midpoint of side AB. Point I is the center of the inscribed circle of triangle ABC (the circle lying inside triangle ABC and tangent to all three of its sides) and point E is the center of the circle outside triangle ABC, and tangent to side AB and to the lines containing sides AB and BC. (This circle is called an *escribed circle.*) Compute the area of quadrilateral $AIME$.



Solution. We find the radii of both circles by matching tangents, since the two tangents from a point to a circle have the same lengths. First, let a, b, c be the respective lengths of the tangents from A, B, C to the inscribed circle. Then

$$a + b = 12$$

$$a + c = 5$$

$$b + c = 13$$

Adding the first and second equations together, $2a + b + c = 17$. Subtracting the third gives $2a = 4$, so $a = 2$. But since the angle at A is right, the quadrilateral formed by A, I, and the two points of tangency is a square. Thus the radius of the inscribed circle is 2.

Similarly, let x, y, z be the respective lengths of the tangents from A, B, C to the escribed circle. Then

$$x + y = 12$$

$$z - x = 5$$

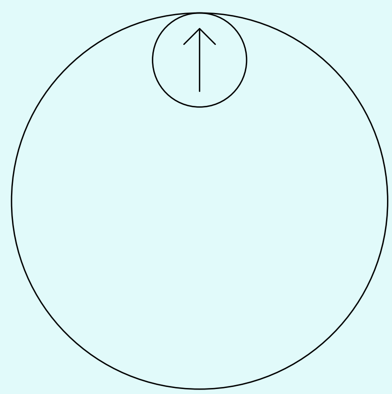
$$z - y = 13$$

Subtracting the second equation from the first gives $2x + y - z = 7$. Adding the third equation to this gives $2x = 20$, so $x = 10$. And again, since there is a right angle at A, the radius of the escribed circle itself must be 10.

Now note that both radii are perpendicular to side AB. So the area of AIM is $\frac{2\cdot 6}{2} = 6$, and the area of AEM is $\frac{10\cdot 6}{2} = 30$. Thus the area of $AIME$ is $\boxed{36}$. ■

Proposed by Matthew Kroesche.
Category: Geometry

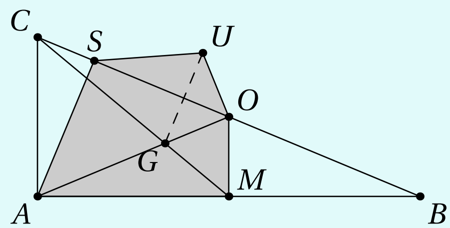
Problem G4. A circle of radius 3 with an arrow facing up drawn on it is internally tangent to another circle with radius 12. Let the inner circle roll along the inside the outer circle. When the arrow faces up for the first time after it starts rolling, what is the degree measure of the arc of the big circle that it has rolled along?



Solution. Suppose the small circle has rolled through an angle of d degrees, so that the point of tangency with the big circle has traced out an arc measuring $12 \cdot \frac{\pi d}{180} = \frac{\pi d}{15}$. Now the small circle has rotated this same length, relative to the curved surface of the big circle, but its radius, and thus, circumference, is four times smaller. Thus it has itself rotated through an angle of $4d$ degrees. But be careful! If the small circle is rolling clockwise, note that the arrow is turning counterclockwise. The counterclockwise rotation of $4d$ degrees is being hampered by the slower, clockwise rotation of d degrees around the clock. So really, relative to a fixed point (say, the center of the circle) the arrow has only rotated $3d$ degrees. Thus, for it to have made a full revolution of 360 degrees, it must have moved along an arc of $\boxed{120}$ degrees. ■

Proposed by Joshua Pate.
Category: Geo
Remark: Adapted from the infamous clockblock problem.

Problem G5. Triangle ABC has $AB = 12$, $AC = 5$ and $BC = 13$, and M is the midpoint of side AB. Point O is the midpoint of side BC, and point S is the point on side BC such that $AS \perp BC$. Point G is the intersection of segments AO and CM, and point U is the reflection of point G over line BC. Compute the area of pentagon $USAMO$.



Solution. First we find the area of quadrilateral $SAMO$ by subtracting the areas of three right triangles:

$$[SAMO] = [ABC] - [SAC] - [MBO]$$

Now $[ABC] = \frac{5\cdot 12}{2} = 30$. Since $SAC \sim ABC$, we have $SC = 5 \cdot \frac{5}{13} = \frac{25}{13}$, and $AS = 5 \cdot \frac{12}{13} = \frac{60}{13}$. So $[SAC] = \frac{1}{2} \cdot \frac{25}{13} \cdot \frac{60}{13} = \frac{750}{169}$. And since $MBO \sim ABC$ and has half the side lengths, its area is $\frac{30}{4} = \frac{15}{2}$. Thus

$$[SAMO] = 30 - \frac{750}{169} - \frac{15}{2} = \frac{10140 - 1500 - 2535}{338} = \frac{6105}{338}$$

Next we find the area of triangle USO. Now $USO \cong GSO$. And since G is the centroid of the triangle, we have $AO = 3GO$, so the altitude from A to BC is three times as long as the altitude from A to BC. But the altitude from A to BC is exactly AS. So in fact we have

$$[USO] = [GSO] = \frac{1}{3}[ASO] = \frac{1}{6} \cdot AS \cdot SO = \frac{1}{6} \cdot \frac{60}{13} \cdot \left(\frac{13}{2} - \frac{25}{13}\right) = \frac{10}{13} \cdot \frac{119}{26} = \frac{595}{169}$$

Thus

$$[USAMO] = [USO] + [SAMO] = \frac{6105}{338} + \frac{595}{169} = \frac{6105 + 1190}{338} = \boxed{\frac{7295}{338}}$$

■

Proposed by Matthew Kroesche.
Category: Geometry

Problem N1. Name goes to random.org and generates a random number n from 50 to 99 (inclusive), and takes the sum of all the prime factors to the maximum power which still divides n. Name sees that this sum is an astoundingly low 14. Find the least possible value of n.

Solution. The only primes or prime powers less than 14 are 2, 3, 4, 5, 7, 9, 11, and 13. 13 obviously cannot be a prime factor of n because $14 - 13 = 1$ is not a prime. Also, 11 cannot be a prime factor of n because $11 * 3 < 50$. 9 also doesn't work because $9 * 5 < 50$ and 9 and 3 cannot both be in the summation. 7 works with either 5 and 2, or 4 and 3, and obviously nothing else. All other possibilities must include at least one of the aforementioned cases because the lowest possible sum without any of them is $5 + 4 + 3 < 14$. Thus, the lowest possible value is $7(5)(2) = \boxed{70} < 7(4)(3)$. ■

Remark: should it be how many possible values are there for n instead?
Proposed by Eli Meyers.
Category: NT(?)

Problem N2. What is 2 more than 2 times the number of divisors of the answer to this problem?

Solution. This is the same as saying what number is equal to two more than twice its number of divisors. Any such number must be less than or equal to 18, since any largish number has at most $n/3+2$ divisors, so twice that plus 2 is $2n/3+6$. Hence, going through the possibilities from 1 to 18 gives that $\boxed{10}$ is the only answer. ■

Proposed by Pierce Lai.
Category: NT

Problem N3. Cardigan and Backyardigan are trying to solve a math problem that reads "Compute the number of positive integer divisors of a^{b^c}.", where a, b, c are some fixed positive integers. Cardigan solves the problem correctly by reading the iterated exponent as $a^{(b^c)}$, and gets an answer of 7400. Backyardigan, however, incorrectly reads the iterated exponent as $(a^b)^c$, but makes no mistakes other than that, and gets an answer of 645. Given this information, compute b^c.

Solution. Let $p_1^{e_1}\cdots p_k^{e_k}$ be the prime factorization of a. Then we have

$$(b^c e_1+1)\cdots(b^c e_k+1) = 7400 = 2^3\cdot 5^2\cdot 37$$

$$(bce_1+1)\cdots(bce_k+1) = 645 = 3\cdot 5\cdot 43$$

The 645 has much fewer cases, so we split it up. Now we note that neither of b, c is one – if $c=1$, then both of their answers would have been the same; and if $c>1$ but $b=1$, then Backyardigan's answer would have been bigger. Also, we cannot have $b=c=2$ or they would both have the same answer. Thus $bc\geq 6$ and bc is not prime. Now each factor on the left must be one of the eight divisors of 645. We rule out $1, 3, 5$ as possibilities since bc would be too small in that case. But the only ways to split 645 up into divisors without using any of these numbers are by either writing it as 645 itself, or as $15\cdot 43$. If we write it as 645, so that $a=p_1^{e_1}$ is a prime power, then we have $bce_1 = 644$ and $b^c e_1 = 7399$. Then b divides the GCD of $644 = 2^2\cdot 7\cdot 23$ and $7399 = 7^2\cdot 151$ which is 7, so it must be that $b=7$ and $c=2$. But this does not give consistent values for e_1, as we get $e_1 = 46$ and $e_1 = 151$ simultaneously. Thus there cannot be just one prime factor of a; there must be two. Thus we have

$$bce_1+1 = 15$$

$$bce_2+1 = 43$$

So $bce_1 = 14$ and $bce_2 = 42$. Since bc divides 14 and is not prime or 1, it must be that $bc = 14$. Then $e_1 = 1$ and $e_2 = 3$. Now either $b=2, c=7$; or $b=7, c=2$. If $b=2, c=7$, then we have

$$(128+1)(128\cdot 3+1) = 7400$$

which is not true. If, however, $b=7, c=2$, then we have

$$(49+1)(49\cdot 3+1) = 7400$$

which is true. So $b=7, c=2$, and $b^c = \boxed{49}$. ■

Proposed by Matthew Kroesche.
Category: NT

Problem N4. Let $p=1997$ be a prime number. The pair of integers $(A,B) = (1,412)$ has the remarkable property that A^2+B^2 is divisible by p. There exist positive integers a and b such that $a^2+b^2 = p$. Compute $a+b$.

Solution. The correct view of this problem is to forget that $a, b > 0$ and instead find sufficiently small a and b such that

$$b \equiv \pm ak \pmod{p}.$$

This will give $p \mid a^2 + b^2$ and hopefully $a^2 + b^2 = p$ if a and b are small enough. We fix the + sign in the above to decrease headaches. Currently, we have two pairs $(0, 1997)$ and $(1, 412)$ which satisfy this equivalence.

Note that directly adding two pairs satisfying $b \equiv ak$ will preserve the equivalence, and clever adding will let us decrease the size of the pairs. Currently $(0, 1997)$ is our largest pair, but

$$(0, 1997) + -5 \cdot (1, 412) = (-5, 1997 - 2060) = (-5, -63).$$

We note $63^2 > 1997$ is still too large. Having dealt with $(0, 1997)$, the pair $(1, 412)$ is still very large, but

$$(1, 412) + 6 \cdot (-5, -63) = (1 - 30, 412 - 378) = (-29, 34).$$

This is a reasonbly small pair, and indeed we can check that $(-29)^2 + 34^2 = 841 + 1156 = 1997$. So our answer is $29 + 34 = \boxed{73}$. ■

Proposed by Nir Elber.

Category: NT

Remark: The trick we used is called (Gaussian) lattice-basis reduction. This problem is almost definitely not suitable for the written contest; decreasing p might help, but someone will have to convince me that doing so will help before I consider it.

Problem N5. Let $d'(n)$ be the number of non-trivial divisors (i.e., divisors other than 1 or n) of the positive integer n, and let S be the set of all positive integers which are not divisible by any primes greater than 5. Find the sum of $\frac{d'(n)}{n}$ over all elements n of the set S.

Solution. Suppose $n = 2^x 3^y 5^z$, where x, y, z are nonnegative integers. Then $d'(n) = (x+1)(y+1)(z+1) - 2$ *unless* $n = 1$, in which case $d'(1) = 0$, not -1. So we want to find the sum of $\frac{(x+1)(y+1)(z+1)-2}{2^x 3^y 5^z}$ over all triples (x, y, z) of nonnegative integers. First we just find the sum of the $\frac{2}{2^x 3^y 5^z}$ – we'll subtract this at the end. We can split this sum up using the distributive law to get a product of three geometric progressions:

$$2\left(1 + \frac{1}{2} + \frac{1}{4} + \cdots\right)\left(1 + \frac{1}{3} + \frac{1}{9} + \cdots\right)\left(1 + \frac{1}{5} + \frac{1}{25} + \cdots\right) = 2 \cdot \frac{1}{1 - \frac{1}{2}} \cdot \frac{1}{1 - \frac{1}{3}} \cdot \frac{1}{1 - \frac{1}{5}} = 2 \cdot 2 \cdot \frac{3}{2} \cdot \frac{5}{4} = \frac{15}{2}$$

As for the other sum, the sum of the $\frac{(x+1)(y+1)(z+1)}{2^x 3^y 5^z}$, we can also split it up using the distributive law to get

$$\left(1 + \frac{2}{2} + \frac{3}{4} + \frac{4}{8} + \cdots\right)\left(1 + \frac{2}{3} + \frac{3}{9} + \frac{4}{27} + \cdots\right)\left(1 + \frac{2}{5} + \frac{3}{25} + \frac{4}{125} + \cdots\right)$$

These three series are not exactly geometric, but they can be "tweaked" to represent them in terms of geometric series. In general if we have the series

$$1 + 2r + 3r^2 + 4r^3 + \cdots$$

we can write it as

$$1 + (r + r) + (r^2 + r^2 + r^2) + (r^3 + r^3 + r^3 + r^3) + \cdots$$

Then we can rearrange these to get a sum of geometric progressions:

$$(1 + r + r^2 + r^3 + \cdots) + (r + r^2 + r^3 + \cdots) + (r^2 + r^3 + \cdots) + \cdots$$

This evaluates to

$$\frac{1}{1-r} + \frac{r}{1-r} + \frac{r^2}{1-r} + \frac{r^3}{1-r}$$

which is itself a geometric progression that sums to

$$\frac{\frac{1}{1-r}}{1-r} = \frac{1}{(1-r)^2}$$

Thus the sum on the left is

$$\frac{1}{\left(1-\frac{1}{2}\right)^2} \cdot \frac{1}{\left(1-\frac{1}{3}\right)^2} \cdot \frac{1}{\left(1-\frac{1}{5}\right)^2} = 4 \cdot \frac{9}{4} \cdot \frac{25}{16} = \frac{225}{16}$$

Subtracting the second sum from the first, we get

$$\frac{225}{16} - \frac{15}{2} = \frac{225-120}{16} = \frac{105}{16}$$

We also have to add 1 to this since our formula assumed $d'(1) = -1$ instead of $d'(1) = 0$. So the sum is $\boxed{\frac{121}{16}}$. ■

Proposed by Joshua Pate.
Category: NT

Problem Who Even Knows What This Is 0. The product of the areas of all non-degenerate triangles with integer side lengths a, b, and c for which $a, b, c \leq 10$ and the solutions to the equation $ax^2 + bx + c = 0$ are all rational can be expressed as a common fraction in simplest radical form as $\frac{r\sqrt{s}}{t}$. What is the value of s?

Solution. Do casework on the discriminant $b^2 - 4ac$. There are a total of 6 ordered triples that satisfy the conditions: $(2,9,9), (3,10,8), (2,7,6), (2,9,10), (1,4,4), (2,8,8)$. Using Heron's Theorem on all 6 triangles, the radical part of the area is $\boxed{187}$. ■

Proposed by Jeffrey Huang.
Category: Algebra, Combo (Casework), NT, Geometry