2022 AMC Practice MATHCOUNTS Solutions Manual

Austin Math Circle

January 9, 2022

Contents

Sprint Problems

Sprint 1. What is the value of 111×123?

Solution. The answer is tediously computed to be $\boxed{13653}$. ■

Proposed by Matthew Kroesche.

Sprint 2. Powell and Donovan have a selenium cube measuring 14 feet on each side. What is the volume of the cube in cubic feet?

Solution. The volume of a cube is the cube of the side length, which in cubic feed is $14^3 = 14^2 \cdot 14 = 196 \cdot 14 = 200 \cdot 14 - 4 \cdot 14 = 2744$. So our answer is $\boxed{2744}\ \mathrm{ft}^3$. ■

Proposed by Joshua Pate.
Remark: I hope someone gets this.

Sprint 3. Suppose $f(x)$ is a quadratic polynomial with integer coefficients, such that $f(1) = 2$, $f(2) = 5$, and $f(3) = 10$. What is $f(4)$?

Solution. By observation, we see that $f(x) = x^2 + 1$, so $f(4) = \boxed{17}$. ■

Proposed by Pierce Lai.

Sprint 4. Suppose that to deep-fry a chicken you need one gallon of oil, and that every additional chicken requires an additional half a gallon of oil. How many chickens can you deep-fry with 10 gallons of oil?

Solution. We see that the answer is $1 + 2 * 9 = \boxed{19}$ chickens. ■

Proposed by Pierce Lai.

Sprint 5. (Insert your name here) is doing the prints round of the competition Cathmounts. There are 40 questions to be done in 30 minutes, each on obscure trivia about typography. If you have a 40% chance of getting any particular question correct, what is the expected number of total questions you get right?

Solution. We see that the answer is $40 * 0.4 = \boxed{16}$ questions. ■

Proposed by Pierce Lai.

Sprint 6. What is $\frac{55^2+1}{17}$?

Solution. We compute $\frac{55^2+1}{17} = \frac{3025+1}{17} = \frac{3026}{17} = \boxed{178}$. ■

Proposed by Joshua Pate.

Sprint 7. Assume a perfectly spherical cow. If a cow has height 6 meters and its volume is $a\pi$ meters cubed, what is the value of a?

Solution. The cow has radius 3 meters, so the answer is $a = 4/3 * 3^3 = \boxed{36}$. ■

Proposed by Pierce Lai.

Sprint 8. Fried, Nugget, and Gizzard are eating a pie. Fried and Nugget together eat twice as much as Gizzard alone, and Fried and Gizzard eat three times as much as Nugget alone. If they eat the entire pie, then what fraction of the pie did Fried eat? Express your answer as a common fraction.

Solution. From the first condition, we know that Gizzard eats $\frac{1}{3}$ of the pie, and the second tells us that Nugget eats $\frac{1}{4}$ of the pie. Hence, Fried eats $1 - \frac{1}{3} - \frac{1}{4} = \boxed{\frac{5}{12}}$ of the pie. ■

Proposed by Pierce Lai.

Sprint 9. Alice finds a pond full of goldfish. Each day, she adds a shark to the pond. Each shark eats one goldfish the first day, two the second day, and n on the nth day. On August 15th, the sharks collectively eat 45 goldfish, leaving just 1 in the pond. How many goldfish were there initially?

Solution. $45 = 1 + \ldots + 10$, so August 15th was the 10th day. The total number of goldfish eaten on the nth day is $1 + \ldots + n$: so the total number of eaten goldfish was $1 + 3 + 6 + 10 + 15 + 21 + 28 + 36 + 45 = 165$, plus the one that wasn't eaten makes a total of $\boxed{166}$. ■

Proposed by Maeve Dever.

Sprint 10. The mean of eleven positive integers is 11, and their unique mode is 10. What is the least possible value of their median?

Solution. Since we can do 3 1s, 3 2s, 4 10s, and some arbitrarily large last number, the least possible value of the median is $\boxed{2}$. ■

Proposed by Pierce Lai.

Sprint 11. Bob has perfectly cubical onions. If a large onion's volume is exactly 2299968 times as large as a small onion's and and the ratio of their widths is an integer, how many times as wide is a large onion compared to a small onion?

Solution. We want to find the cube root of 2299968. We see that $130^3 = 2197000$, so the answer is probably a little bit larger than that. Looking at the units digit tells us the answer is $\boxed{132}$. ■

Proposed by Pierce Lai.

Sprint 12. What is the largest positive integer n such that 3^n divides $((3!)!)!$?

Solution. $((3!)!)! = 720!$, which has $240 + 80 + 26 + 8 + 2 = \boxed{356}$ 3s. ■

Proposed by Pierce Lai.

Sprint 13. Alex is watching anime. Each season of anime consists of 13 episodes, and each episode takes 20 minutes to watch. What is the number of whole days he needs to watch 100 seasons of his favorite anime, assuming he does not need to take any breaks?

Solution. Each season of anime takes $4\frac{1}{3}$ hours, so 100 seasons is $433\frac{1}{3}$ hours. We see that this is just over $18 * 24 = 432$ hours, so the answer is $\boxed{19}$ days. ■

Proposed by Pierce Lai.

Sprint 14. Let a prime p be a Chen prime if $p+2$ is either a prime or a product of two (not necessarily distinct) primes. What is the second prime that is not a Chen prime?

Solution. Because the numbers are small, it will be faster to examine positive integers n which are neither prime nor to the product of two primes and then ask if $n-2$ is a prime. Because $n \geq 4$, we see that $n-2$ will be odd, so if $n-2$ is to be prime, then n must be odd. In particular, we see that n is the product of at least three primes, all of which must be odd. We now to casework.

- We see $3 \cdot 3 \cdot 3 - 2 = 25$ is not a prime.
- We see $3 \cdot 3 \cdot 5 - 2 = 43$ is a prime. So 43 is the first prime which is not a Chen prime.
- We see $3 \cdot 3 \cdot 7 - 2 = 61$ is a prime. So $\boxed{61}$ is the second prime which is not a Chen prime.

For completeness, we note that $3 \cdot 5 \cdot 5 - 2 = 73$ is larger than 61, so 61 remains the second prime which is not a Chen prime. ■

Proposed by Joshua Pate.

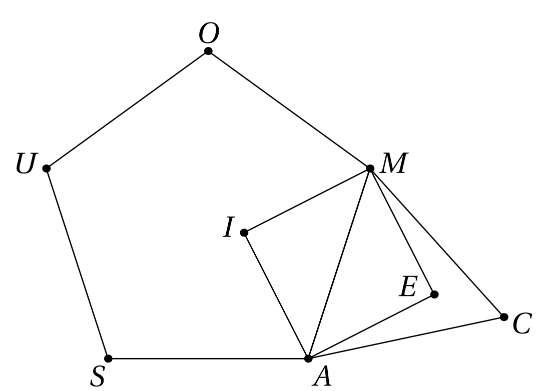
Sprint 15. A frog with a wooden cane sits at the bottom of a staircase. It can jump up either 1 step or 3 steps every jump. If the staircase has 10 steps, how many sequences of jumps can it take to land exactly on the top of the staircase?

Solution. If we write out the number of possible sequences for steps from 0-10, they are 1, 1, 1, 2, 3, 4, 6, 9, 13, 19, 28. (Every term s_n in the sequence is equal to $s_{n-1} + s_{n-3}$, for $n \geq 3$.) Hence, the answer is $\boxed{28}$ sequences . ■

Proposed by Pierce Lai.

Sprint 16. Let AMC, $AIME$, and $USAMO$ be regular polygons where C and E are not inside $USAMO$. Compute $\angle IMO + \angle EMC$ (in degrees).

Solution. We have the following diagram.



Note that the interior angles of the regular pentagon $USAMO$ have measure $180^\circ - \frac{360^\circ}{5} = 108^\circ$. It follows that

$$\angle IMO = \angle AMO - \angle AMI = 108^\circ - 45^\circ,$$

where $\angle AMI = 45^\circ$ because $\overline{AM}$ is a diagonal of square $AIME$. Similarly, we see that

$$\angle EMC = \angle AMC - \angle AME = 60^\circ - 45^\circ,$$

where $\angle AME = 45^\circ$ for the same reason. In total, we see that

$$\angle IMO + \angle EMC = 108^\circ + 60^\circ - 45^\circ - 45^\circ = 168^\circ - 90^\circ = \boxed{78^\circ},$$

which is what we wanted. ■

Proposed by Joshua Pate.

Sprint 17. Madeline has a octahedron-shaped strawberry inscribed in a sphere, which is itself inscribed in a cube. What is the ratio of the volume of the strawberry to the volume of the cube? Express your answer as a common fraction.

Solution. To make things easier, rotate the octahedron so that its vertices are precisely the centers of the faces of the cube. Then, if we chop the two polyhedra in half parallel to a face of the cube, we get a square pyramid and a half-cube. The square pyramid has a base half the size of the base of the half-cube and a height equal to the half-cube, so the answer is $\frac{1}{2} * \frac{1}{3} = \boxed{\dfrac{1}{6}}$. ■

Proposed by Pierce Lai.

Sprint 18. Let a triangular number be a number of the form $\frac{n(n+1)}{2}$, and let a generalized pentagonal number be a number of the form $\frac{n(3n-1)}{2}$ (for integers n). What is the third nonnegative integer that is both a triangular number and a generalized pentagonal number?

Solution. We tabulate, noting that negative values of n are permitted. Here are the first few pentagonal numbers.

n	0	1	−1	2	−2	3	−3	4	−4
$n(3n-1)/2$	0	1	2	5	7	12	15	22	26

And here are the first few triangular numbers. Observe that $\frac{(-n)(-n+1)}{2} = \frac{n(n-1)}{2} = \frac{(n+1)(n+1-1)}{2}$, so there is no need to consider negative indices for triangular numbers.

n	0	1	2	3	4	5	6
$n(3n-1)/2$	0	1	3	6	10	15	21

We see that the matching nonnegative integers are, in order, $0, 1, \boxed{15}$. ■

Proposed by Joshua Pate.

Sprint 19. Let $(1+\sqrt{5})^4 = a + b\sqrt{5}$, where a and b are integers. Find $a+b$.

Solution. We proceed by direct expansion. Note that

$$\begin{aligned}\left(1+\sqrt{5}\right)^2 &= 1+2\sqrt{5}+5\\ &= 6+2\sqrt{5},\\ \left(1+\sqrt{5}\right)^4 &= \left(6+2\sqrt{5}\right)^2\\ &= 36+24\sqrt{5}+20\\ &= 56+24\sqrt{5}.\end{aligned}$$

Thus, our answer is $56+24=\boxed{80}$. ■

Proposed by Joshua Pate.

Sprint 20. Bonzu Pippinpaddleopsicopolis rolls a fair six-sided die, a fair eight-sided die, a fair twelve-sided die, and a fair twenty-sided die. What is the probability that the six-sided die is showing a strictly bigger number than all three of the numbers on the other dice? Express your answer as a common fraction.

Solution. Do casework on what the number on the six-sided die is. If it's equal to n, then there are $(n-1)^3$ ways to choose the numbers on the other three dice so that they're all smaller. Thus the answer is

$$\frac{5^3+4^3+3^3+2^3+1^3+0^3}{6\cdot 8\cdot 12\cdot 20} = \frac{15^2}{2^8\cdot 3^2\cdot 5} = \boxed{\frac{5}{256}}$$

■

Proposed by Matthew Kroesche.

Sprint 21. Suppose that $a^3+b^3=4104$ for positive integers a and b. Find the sum of the possible values for ab.

Solution. The main idea is that one of $\{a,b\}$ needs to have its cube between $\frac{4104}{2}=2052$ and 4104; without loss of generality, $2052\le b^3<4104$ so that $\sqrt[3]{2052}\le b<\sqrt[3]{4104}$. By some computation, we can achieve the bounds

$$12^3=1728<2052<2197=13^3 \qquad \text{and} \qquad 16^3=4096<4104<4096+16=16^2\cdot 17<17^3.$$

Thus, $13\le b\le 16$.

To optimize which b we have to actually check, we note that the cubes $\pmod 7$ are in $\{0,1,6\}$ while $4104=4200-98+2\equiv 2 \pmod 7$, so the only way for $a^3+b^3\equiv 2 \pmod 7$ is for $a^3\equiv b^3\equiv 1 \pmod 7$. This means that $b \pmod 7 \in \{1,2,4\}$.

So, among $13\le b\le 16$, the only candidates are $b=15\equiv 1 \pmod 7$ and $b=16\equiv 2 \pmod 7$. From these we see that

$$\sqrt[3]{4104-15^3} = \sqrt[3]{9\left(456-3\cdot 5^3\right)} = 3\sqrt[3]{152-125} = 3\cdot 3 = 9,$$

so $9^3+15^3=4104$. Similarly,

$$\sqrt[3]{4104-16^3} = \sqrt[3]{4104-4096} = \sqrt[3]{8} = 2,$$

so $2^3+16^3=4104$. So the possible pairs (a,b) are $(9,15)$ and $(2,16)$ and reflections, so the possible values of ab are 135 and 32, which sum to $135+32=\boxed{167}$. This finishes. ■

Proposed by Joshua Pate.

Sprint 22. Let a Conway number be a composite positive integer that is not divisible by any of 2, 3, 5, or 11; and is not a perfect square. What is the third Conway number?

Solution. We see that the numbers of interest must prime factors in $\{7, 13, 17, 19, \ldots\}$. Because perfect squares are not permitted, we can list out the first few Conway numbers as $7 \cdot 13 = 91$ and $7 \cdot 17 = 119$ and $7 \cdot 19 = 133$. Note that $13 \cdot 17 = 221$ is too big, so we see that, indeed, $\boxed{133}$ is the third Conway number. ■

Proposed by Joshua Pate.
Remark: https://xkcd.com/2293/

Sprint 23. Let A be the base 10 integer which is written in base 3 as 2022_3. Let B_3 be the base 3 representation of the base 10 integer 2022_{10}, and let C be the product (in base 10) of the nonzero digits of B_3. Compute $A + C$ (in base 10).

Solution. We see that

$$A = 2022_3 = 2 \cdot 3^3 + 0 \cdot 3^2 + 2 \cdot 3 + 2 = 54 + 6 + 2 = 62.$$

Then we can also compute the base-three representation of 2022 by repeated division, noting

$$\begin{aligned}
\frac{2022}{3} &= 674 + \frac{0}{3} \\
\frac{674}{3} &= 224 + \frac{2}{3} \\
\frac{224}{3} &= 74 + \frac{2}{3} \\
\frac{74}{3} &= 24 + \frac{2}{3} \\
\frac{24}{3} &= 8 + \frac{0}{3} \\
\frac{8}{3} &= 2 + \frac{2}{3} \\
\frac{2}{3} &= 0 + \frac{2}{3}.
\end{aligned}$$

So we can read off $2022_{10} = 2202220_3$, so the product of the nonzero digits is $C = 32$. So in total, we find that $A + C = \boxed{94}$. ■

Proposed by Joshua Pate.

Sprint 24. The numerical values of the length, width, and area of a rectangle form an increasing arithmetic progression in this order. If the area of the rectangle is 4, what is its perimeter? Express your answer in simplest radical form.

Solution. Letting ℓ, w denote the length and width respectively, we have

$$\ell w = 2w - \ell = 4$$

We write $w = \frac{4}{\ell}$ and thus

$$\frac{8}{\ell} - \ell = 4$$

$$\ell^2 + 4\ell - 8 = 0$$

$$\ell = -2 + 2\sqrt{3}$$

since we obviously have $\ell > 0$. Then

$$w = \frac{\ell + 4}{2} = 1 + \sqrt{3}$$

and so the perimeter is

$$2(\ell + w) = \boxed{6\sqrt{3} - 2}$$

■

Proposed by Matthew Kroesche.

Sprint 25. How many positive perfect squares have five or fewer digits, and have a 1, 2, or 3 as their leftmost digit?

Solution. The number of n-digit perfect squares whose leftmost digit is a 1, 2, or 3 is just the number of perfect squares in $[10^{n-1}, 4 \times 10^{n-1})$, and this is

$$\left\lceil \sqrt{4 \times 10^{n-1}} \right\rceil - \left\lceil \sqrt{10^{n-1}} \right\rceil$$

If n is odd, both square roots are integers so we have

$$2 \cdot 10^{\frac{n-1}{2}} - 10^{\frac{n-1}{2}} = 10^{\frac{n-1}{2}}$$

So for $n = 1, 3, 5$ we have $1, 10, 100$ possible solutions respectively. If n is even, we have

$$\left\lceil 2 \cdot 10^{\frac{n}{2}-1} \sqrt{10} \right\rceil - \left\lceil 10^{\frac{n}{2}-1} \sqrt{10} \right\rceil$$

So for $n = 2$ we have

$$\left\lceil 2\sqrt{10} \right\rceil - \left\lceil \sqrt{10} \right\rceil = 7 - 4 = 3$$

and for $n = 4$ we have

$$\left\lceil 20\sqrt{10} \right\rceil - \left\lceil 10\sqrt{10} \right\rceil = 64 - 32 = 32$$

(We can either use crude approximations like $\sqrt{10} \approx 3.16$, or just use trial and error; e.g. for four-digit squares it's not too tedious to compute that $32^2 = 1024$ is the smallest and $63^2 = 3969$ is the largest.) Thus, summing these up gives $1 + 3 + 10 + 32 + 100 = \boxed{146}$. ■

Proposed by Matthew Kroesche.

Sprint 26. Let $2\sqrt{2\sqrt[3]{2\sqrt{2\sqrt[3]{2\ldots}}}} = 2^n$. What is n? Express your answer as a common fraction.

Solution. Observe that

$$\begin{aligned}
2^n &= 2\sqrt{2\sqrt[3]{2\sqrt{2\sqrt[3]{2\ldots}}}} \\
&= 2\sqrt{2\sqrt[3]{2\sqrt{2\sqrt[3]{2\ldots}}}} \\
&= 2\sqrt{2\sqrt[3]{2^n}} \\
&= 2 \cdot 2^{1/2} \cdot \left(2^n\right)^{1/6}.
\end{aligned}$$

Thus, $n = 1 + \frac{1}{2} + \frac{n}{6}$, which rearranges into $6n = 6 + 3 + n$, which gives $n = \boxed{\frac{9}{5}}$. ■

Proposed by Joshua Pate.

Sprint 27. Given that the number

$$N = \frac{20^{22} - 22}{20 + 22}$$

is an integer, how many digits long is it?

Solution. We write this as

$$N = \frac{20^{22} + 20}{42} - 1 = \frac{10}{21}\left(2^{21}10^{21} + 1\right) - 1$$

Now 2^{10} is just a little bigger than 1000, so 2^{21} is just a little bigger than 2000000. Thus the number $2^{21}10^{21} + 1$ is just a little bigger than 2×10^{27}. We unfortunately need to know if it's bigger than 2.1×10^{27}, since that will influence whether N has 28 digits or only 27. Thus the question is to determine if 2^{21} is bigger than 2.1×10^6. We determine that it is not, since $2^{10} = 10^3 + 24$, so $2^{20} = 10^6 + 2 \times 10^3 \times 24 + 24^2 = 1048576 < 1050000$. Thus $N < 10^{27}$ so N has only $\boxed{27}$ digits. ■

Proposed by Matthew Kroesche.

Sprint 28. Let $ABCD$ be the vertices on a regular tetrahedron. Assume that there are spheres with radius equal to half the side length of the tetrahedron centered around each of A, B, C, D. What is the ratio of radius of the sphere that is internally tangent to each of the other spheres to the sphere that is externally tangent to each of the other spheres? Express your answer in simplest radical form.

Solution. Give $ABCD$ a side-length of s, for concreteness. Let O be the center of the tetrahedron; by symmetry, O will be the center of both the sphere internally tangent to the other spheres as well as the center of the sphere externally tangent to the other spheres. Let the radii of the two spheres centered at O be R and r, respectively, so that we are interested in computing R/r.

Let S_A be the sphere of radius $\frac{1}{2}$ centered at A. Let the intersection of $\overrightarrow{OA}$ with S_A inside of $ABCD$ be X and the intersection outside of $ABCD$ be Y. Then

$$r = OX = OA - AX = OA - \frac{s}{2} \qquad \text{and} \qquad R = OY = OA + AY = OA + \frac{s}{2}.$$

So we are interested in computing OA.

To optimize our work, we pull a trick: we scale the tetrahedron and embed it into $\mathbb{R}^3$ so that $A = (1,1,1)$ and $B = (1,-1,-1)$ and $C = (-1,1,-1)$ and $D = (-1,-1,1)$. We see that all side lengths of $ABCD$ are $s = 2\sqrt{2}$, and by symmetry $O = (0,0,0)$. And further, we can see that $OA = \sqrt{3}$, so the desired ratio is

$$\frac{R}{r} = \frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}} = \frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}} \cdot \frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}+\sqrt{2}} = \boxed{5+2\sqrt{6}}.$$

This is what we wanted. ■

Proposed by Joshua Pate.

Sprint 29. Let f be a function mapping integers to integers such that for all integers x and $k \geq 0$, $f(x+2^k) - f(x)$ is an integral power of 2. Given that $f(1) + f(2) = 10$ and $f(2020) + f(2021) = 16162$, what is $f(2022)$?

Solution. Taking $k = 0$, we know that $f(x+1) - f(x)$ and $f(x+2) - f(x+1)$ are powers of 2. We also know that $f(x+2) - f(x)$ is a power of 2. Thus we have two powers of 2 adding to another power of 2 - consider the equation $2^a + 2^b = 2^c$. Without loss of generality, assume $a \leq b$. Dividing both sides by 2^a gives $1 + 2^{b-a} = 2^{c-a}$. Since the RHS is even, we require 2^{b-a} to be odd. Thus we can only have $2^{b-a} = 1 \implies b = a$. In particular, $f(x+1) - f(x) = f(x+2) - f(x+1)$.

Using this for all x tells us that f actually produces an arithmetic progression. It's not hard to confirm that this arithmetic progression satisfies the problem assertion.

Now write $f(1) = a + d$, $f(2) = a + 2d$, $f(2020) = a + 2020d$, $f(2021) = a + 2021d$, which we are allowed to do because of the arithmetic progression. Plugging these in gives $2a + 3d = 10$, $2a + 4041d = 16162$. Subtracting gives $4038d = 16152 \implies d = 4$, so $a = -1$. Finally, $f(2022) = a + 2022d = \boxed{8087}$. ■

Proposed by Arnav Adhikari.

Sprint 30. In acute triangle ABC, the feet of the altitudes from A, B, C to the opposite sides are R, S, T respectively; and the three altitudes meet at a point H. Point M is such that S is the midpoint of MR and $MATH$ is a square. What is $\frac{AC}{BC}$? Express your answer as a common fraction in simplest radical form.

Solution. Since $AS \perp HS$, we see that S lies on the circle with diameter AH, which is the circumscribed circle of square $MATH$. Since S is the midpoint of MR, and since R lies on line AH, we see that M is twice as far from line AH as S is. Supposing WLOG that circle $MATH$ has radius 1, we see that M (being the midpoint of semicircular arc AMH) is a distance of 1 from line AH, so S is a distance of $\frac{1}{2}$ from it. Since S is also a distance of 1 from O (O being the center of square $MATH$) we have that $\angle SOH = 30°$ and thus $\angle SAH = 15°$. This, together with $\angle TAH = 45°$ (since $MATH$ is a square) helps us determine all the angles in triangle ABC. We have $\angle BAC = \angle SAH + \angle TAH = 60°$, $\angle ABC = 90° - \angle TAH = 45°$, and $\angle ACB = 90° - \angle SAH = 75°$. Then we can finish by using what we know about 30-60-90 and 45-45-90 triangles. If $AT = 1$, then $CT = \sqrt{3}$ and $AC = 2$ because CAT is a 30-60-90 triangle. Then $BT = \sqrt{3}$ as well and $BC = \sqrt{6}$ because CBT is a 45-45-90 triangle. So $\frac{AC}{BC} = \frac{2}{\sqrt{6}} = \boxed{\frac{\sqrt{6}}{3}}$. ■

Proposed by Matthew Kroesche.

Target Problems

Target 1. Suppose Alex's favorite number is 42 (the critical angle in degrees of a rainbow), Pierce's favorite number is 69 (as its square and cube use every digit from 0-9), and Matthew's favorite number is 135 (roughly 1% of his age). Suppose the product of their favorite numbers is N. What is the sum of all of the positive divisors of N?

Solution. The prime factorization of N is $2 * 3^5 * 5 * 7 * 23$, so the answer is $3 * 364 * 6 * 8 * 24 = 1257984$. ■

Proposed by Pierce Lai.

Target 2. Pierce likes to listen to regular human music. When he listens to a regular human song, after every 10 seconds (in real time) he regular humanly speeds the song up by 10% multiplicatively. Suppose he starts listening to a song which was originally 5 minutes, and it starts out at 1x speed. When the song ends, if the speed of the song playing is 1.1^a, what is the value of a?

Solution. We see that at the end of the 1.1^a speed interval, a total of $10(\frac{1.1^{a+1}-1}{1.1-1}) = 100(1.1^{a+1} - 1)$ seconds of the song in regular human time will have passed. 5 minutes is 300 seconds, so this turns into $1.1^{a+1} > 4$. We see that $1.1^8 = 1.4641^2 \approx 1.46^2 \approx 1.96 + 0.168 \approx 2.13$. Hence, $1.1^{16} \approx 2.13^2 \approx 4.41 + 0.126 = 4.54$, and so the answer is $a = \boxed{14}$. (With a calculator, $1.1^8 = 2.1436$ and $1.1^{16} = 4.595$.) ■

Proposed by Pierce Lai.

Target 3. There are 2022 people at a party, some of whom shake hands with each other. Kiyo shakes hands with more people than any other guest at the party. If there are exactly 100,000 total handshakes at the party, what is the least number of people Kiyo could possibly have shaken hands with?

Solution. If Kiyo shook hands with N people, then every one of the 2021 other guests shook hands with at most $N-1$ people, for a total of $N+2021(N-1) = 2022N-2021$ handshakes. However, each handshake is participated in by two people, so we must divide this by two. This gives that the total number of handshakes is at most $1011(N-1)$, and this must therefore be greater than or equal to 100000. So $N \geq \frac{100000}{1011} + 1$, and thus we have that N is at least $\boxed{100}$. ■

Proposed by Matthew Kroesche.

Target 4. A regular polygon $\mathscr{P}$ has the property that its interior can be divided into some number of non-overlapping regular polygons, each of which has fewer sides than $\mathscr{P}$. What is the largest number of sides that $\mathscr{P}$ can possibly have?

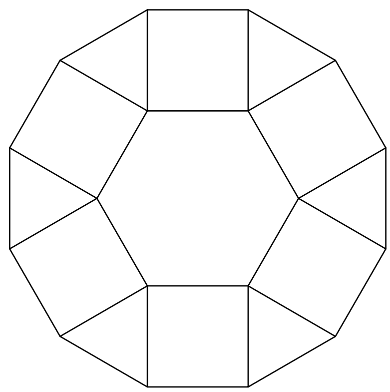
Solution. The main idea is to focus on decomposing an internal angle of $\mathscr{P}$. Namely, once $\mathscr{P}$ has been divided into regular polygons, each internal angle must have been decomposed into the internal angle of other regular polygons. In fact, there must be more than one regular polygon at each vertex of $\mathscr{P}$, for otherwise we would have to use a regular polygon with the same number of sides of $\mathscr{P}$ to fill the corner.

But placing these polygons at a vertex implies that the sum of the internal angles of these regular polygons is less than 180°. Because there are at least two of these polygons, this means that at least one of the regular polygons has internal angle less than 90°, which implies that there must be a triangle at the vertex with internal angle 60°, which is the smallest possible internal angle of a regular polygon.

Because $3 \cdot 60° = 180°$, there must be just two regular polygons at each vertex of $\mathscr{P}$. Running through the possibilities while keeping the internal angles less than 180°, we have the following cases.

- We may have two triangles, giving $60^\circ + 60^\circ = 120^\circ$, which would make $\mathscr{P}$ a hexagon.
- We may have a triangle and a square, giving $60^\circ + 90^\circ = 150^\circ$, which would make $\mathscr{P}$ a dodecagon.
- We may have a triangle and a pentagon, giving $60^\circ + 108^\circ = 168^\circ$, which would make $\mathscr{P}$ a 30-gon (since $\frac{360}{180-168} = 30$. However, this is impossible because the third vertex of an equilateral triangle would have to be shared with two pentagons. (The pentagons must have the same side length as the triangle because otherwise, at least one of them has strictly shorter side length than the triangle [or else they overlap] and thus leaves a 72° angle which cannot be filled by any combination of regular polygons.) Then the remaining angle left is $360^\circ - 108^\circ - 108^\circ - 60^\circ = 84^\circ$, which also cannot be filled by any combination of regular polygons.

With the above in mind, it suffices to actually decompose the dodecagon into regular polygons. Making sure to place a triangle and a square at each vertex reveals the following decomposition.



So indeed, $\boxed{12}$ is the largest number of sides that such a polygon $\mathscr{P}$ could have. ■

Proposed by Joshua Pate.

Target 5. Boib is having twelve guests over, and serves them a plate of twelve jelly-filled donuts. However, he hates three of the guests, and thus has filled three of the donuts with mustard instead of jelly. Boib has unfortunately forgotten which donuts are which. If each of the guests picks randomly from the donuts, what is the probability that the three guests Boib hates will take the three mustard-filled donuts? Express your answer as a common fraction.

Solution. Assume the hated guests pick first. The chance is thus $\frac{1}{4}\frac{2}{11}\frac{1}{10} = \boxed{\frac{1}{220}}$. ■

Proposed by Pierce Lai.

Target 6. Matthew bought three items from the grocery store: a carton of milk, a bag of coffee grounds, and a container of sugar. The day before the coffee was put on the shelves, there were three times as many days until the milk expired as there were after the sugar was put on the shelves, and the day the the milk had 30 days until expiring, the sugar had been out for twice as long as the coffee. What is the product of the numbers of days that the sugar and coffee had been out when the sum of the numbers of days that the sugar and the coffee had been out was equal to the number of days before the milk expired?

Solution. Suppose that Matthew visits the store on day 0. With this reference point, let M be the day that the milk expires, c be the day that the coffee was placed on the shelves, and s be the day that the sugar was placed on the shelves. Note that we are permitting days to be negative and in particular expect $c < 0$ and $s < 0$.

Upon parsing the problem, we have the following information.

(1) On day $c-1$, we have that $M - (c-1) = 3((c-1) - s)$.

(2) On day $M-30$, we have that $(M-30)-s = 2((M-30)-c)$.

(3) On day 0, we have $(0-s)+(0-c) = M-x$.

We can simplify these equations into the system

$$\begin{cases} M+3s-4c=-4, & (1) \\ M+s-2c=-30, & (2) \\ M+s+c=0. & (3) \end{cases}$$

Now we may solve: (3) − (2) gives $3c=-30$, so $c=-10$. Plugging this into (1) and (2) gives $M+s=10$ and $M+3s=-44$, which upon subtracting gives $2s=-54$ and so $s=-27$.

To finish, we see that the problem asks for $(0-s)(0-c) = 27\cdot 10 = \boxed{270}$, so we are done. ■

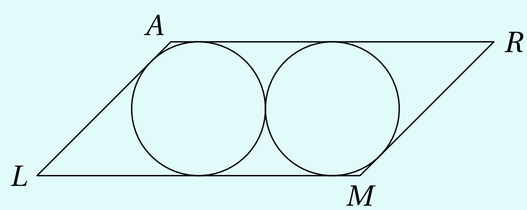
Proposed by Joshua Pate.

Target 7. There are N people playing a very large-scale game. One person is eliminated, and then the remaining people can be divided evenly into groups of 13. Then, four more people are eliminated, and the remaining people can be divided evenly into groups of 11. Finally, one more person is eliminated, and the remaining people can now be divided evenly into groups of 10. What is the least possible value of N such that this is true?

Solution. The key observation here is that $N-6$ is divisible by 5, so $N-1$ is divisible by 5 as well as by 13. Also, $N-6$ is even, so N is as well, and thus N is 1 more than a multiple of 65 and in fact 66 more than a multiple of 130 (since it's also even). We also want $N-5$ to be divisible by 11. Now if $N=66$, we have that N itself is divisible by 11. Adding 130 causes the remainder upon division by 11 to decrease by 2 (since 132 is divisible by 11) so adding it three times causes it to decrease from 0 to -6 (which is equivalent to 5). Thus we answer $130\cdot 3+66 = \boxed{456}$. ■

Proposed by Matthew Kroesche.
Remark: Yes, I am hilarious.

Target 8. In parallelogram $ARML$, circle ω_1 has radius 1 and is tangent to sides $\overline{AR}$, $\overline{AL}$, and $\overline{ML}$, and circle ω_2 has radius 1 and is tangent to sides $\overline{AR}$, $\overline{RM}$, and $\overline{ML}$. Furthermore, circles ω_1 and ω_2 are externally tangent, and circle ω_2 is tangent to side $\overline{AR}$ at its midpoint. What is the value of RL^2? Express your answer in simplest radical form.



Solution. Let $AR = x$, so that the tangents from R to ω_2 have length $\frac{x}{2}$ and the tangents from A to ω_1 have length $\frac{x}{2}-2$, since the distance between the points where ω_1 and ω_2 are tangent to $\overline{AR}$ is 2. Then the tangents from M to ω_2 have length $\frac{x}{2}-2$ as well, since ω_1 and ω_2 have the same radius and $\angle A = \angle M$, so then $RM = \frac{x}{2}+\frac{x}{2}-2 = x-2$. Now, consider trapezoid $RMXY$, where ω_2 is tangent to $\overline{ML}$ at X and $\overline{AR}$ at Y. We have $RM = x-2$, $MX = \frac{x}{2}-2$, $XY = 2$, and $YR = \frac{x}{2}$. Furthermore, $\overline{MX}\perp\overline{XY}$ and $\overline{XY}\perp\overline{YR}$. Thus we apply the Pythagorean Theorem:

$$(YR-MX)^2 + XY^2 = RM^2$$

$$2^2 + 2^2 = (x-2)^2$$

So $x-2 = 2\sqrt{2}$, $x = 2\sqrt{2}+2$, and we see that $\angle R = 45°$ and thus $\angle A = 135°$. Now, drop the perpendicular $\overline{LZ}$ from L to line $\overleftrightarrow{AR}$. We see that $LZ = AZ = 2$, and thus $RZ = AR + AZ = 2\sqrt{2}+4$. So $RL^2 = RZ^2 + LZ^2 = (8+16+16\sqrt{2})+4 = \boxed{28+16\sqrt{2}}$. ■

Proposed by Matthew Kroesche.

Team Problems

Team 1. Suppose $a \odot b = a$ if b equals 0, and $a \odot b = (a^2 + b) \odot (b - 1)$ otherwise. What is the value of $4 \odot 3$?

Solution. We have

$$4 \odot 3 = 19 \odot 2 = 363 \odot 1 = 131770 \odot 0 = \boxed{131770}$$

■

Proposed by Pierce Lai.

Team 2. Vegeta has a lovely bunch of dragon balls. When he's not looking, Frieza steals half of them. Then Zarbon steals five more of them. Finally, Goku takes a third of the remaining dragon balls. When Vegeta looks back, he only has six dragon balls. How many dragon balls did he originally have?

Solution. Work backwards. Goku took away 3 dragon balls, so before he took them, there were 9. Thus before Zarbon took any, there were 14, and so at the beginning there were $14 \times 2 = \boxed{28}$. ■

Proposed by Matthew Kroesche.

Team 3. Call a *fried* number a number which is divisible by exactly 2 distinct primes. Call a *deep-fried* number a number which is divisible by exactly 2 fried numbers. Call a *ultra-deep-fried* number a number which is divisible by exactly 2 deep-fried numbers. Between 1 and 100 inclusive, how many ultra-deep-fried numbers are there?

Solution. A fried number is any number of the form $p^i q^j$, where $i, j \geq 1$ and p, q are distinct primes. Note that any number divisible by 3 primes numbers p, q, r would also be divisible by the three fried numbers pq, qr, and pr; hence, deep-fried numbers can also only be divisible by exactly 2 distinct primes. Thus, deep-fried numbers are numbers of the form $p^2 q$ (divisible by the fried numbers pq and $p^2 q$). Hence, we see that ultra-deep-fried numbers are numbers of the form $p^i q^j$ or $p^i qr$, where i and j are at least 2.

For the first case, we have that $2^2 3^2, 2^3 3^2$, and $2^2 5^2$ are solutions. For the second, we have that $2^2 3^1 5^1, 2^2 3^1 7^1$, and $2^1 3^2 5^1$ are solutions. Hence, the answer is $\boxed{6}$. (Thanks to Matthew for bugcheck.) ■

Proposed by Pierce Lai.

Team 4. Matthew has a whiteboard with the number 2 written on it. He spins a spinner which randomly selects one of the arithmetic operations $+, -, \times, \div$ with equal probability, and replaces the number N on the board with $N \square 3$, where $\square$ is the operation the spinner selected. He repeats this process four more times. What is the expected value of the final number on the board? Express your answer as a common fraction.

Solution. The main idea is that we may pretend the $+$ and $-$ operations do nothing. Namely, for any ordered 5-tuple of operations chosen from $\{+, -, \times, \div\}$, we may exchange each $+$ with a $-$ and a $-$ with a $+$ to produce a different 5-tuple which averages with the first tuple to cancel out the signs. For example, if the spinner gives $(+, \times, -, \div, -)$, then we pair this tuple off with $(-, \times, +, \div, +)$, and these sum to

$$\big(((2+3) \times 3 - 3) \div 3 - 3\big) + \big(((2-3) \times 3 + 3) \div 3 + 3\big) = \big(((2+0) \times 3 - 0) \div 3 - 0\big) + \big(((2-0) \times 3 + 0) \div 3 + 0\big)$$

by the distributive law. With this in mind, we may rephrase the problem so that the spinner has a one-fourth probability of multiplying by 3, one-fourth probability of multiplying by $\frac{1}{3}$, or one-half probability of doing nothing (which is multiplying by 1), and the expected value will not change.

From here, it is not too difficult to see that the sum of all possible outcomes from the spinning is

$$2\left(3+\frac{1}{3}+1+1\right)^5$$

by fully distributing. Thus, the expected value is

$$\frac{2\left(5+\frac{1}{3}\right)^5}{4^5}=\frac{2\cdot 16^5/3^5}{4^5}=\frac{2\cdot 4^5}{3^5}=\boxed{\frac{2048}{243}},$$

which is what we wanted. ■

Proposed by Joshua Pate.

Team 5. Exactly one of the suspects below is lying, and not all are innocent:

Alice: I did not steal the cookies from the cookie jar.
Ballast: If Alice stole the cookies, so did Ellis. Exactly two of us are guilty.
Chalice: If Ballast is lying, Ballast is innocent.
Dallas: If Ballast is lying, I am innocent, and if Ballast is guilty then Ellis is telling the truth.
Ellis: Chalice and I are either both innocent or both guilty.

Who stole the cookies from the cookie jar? (That is, who is guilty? Note that the guilty party may be more than one person.)

Solution. If A is lying, then he did steal the cookies; by (B), E is then also guilty. By (E), if E is guilty, so is C; so A, C and E are all guilty. B says exactly two of them are guilty, so A must not be lying.

If B is lying, then he is innocent by (C); A is innocent by (A); D is innocent because B is lying by (D); so E and C are either both innocent or both guilty. If they are both innocent, all suspects are innocent, which is not allowed. If t hey are both guilty, then B's statement is true: the conditional is vacuously true because A is not guilty, and exactly two suspects are guilty. So, B is not lying.

If C is lying, then B is innocent just in case he is not lying. He isn't, so B is innocent by (C), and A by (A). By (B), exactly two of C, D and E are then guilty; if D is guilty, then either both or neither of C and E is, giving one or three criminals. So, D must be innocent and C and E guilty. This is in fact the only possibility.

If D is lying, then B is not lying, and neither is E; so the first conditional in D's statement is true because B is not lying, and the second is true because E is not lying. So, D's statement must be true.

If E is lying, A is guilty by (C), so E is telling the truth by (D), so E cannot be lying.

Therefore, C is lying, and the criminals are $\boxed{\text{Chalice and Ellis}}$. ■

Proposed by Maeve Dever.

Team 6. The Sphere of Splendor has the same surface area as the Cube of Chaos. The Cube of Chaos has the same volume as the Sphere of Avarice. The Sphere of Avarice has the same surface area as the Cube of Ethan. What is the ratio of the volume of the Cube of Ethan to the volume of the Sphere of Splendor? Express your answer as a common fraction in terms of π.

Solution. We find the ratio of their surface areas first, since this is just the ratio of the surface area of the Sphere of Avarice to the ratio of the surface area of the Cube of Chaos. Now let the Sphere of Avarice have radius r, so its surface area is $4\pi r^2$ and its volume is $\frac{4}{3}\pi r^3$. Then the volume of the Cube of Chaos is also $\frac{4}{3}\pi r^3$, so its side length is $\sqrt[3]{\frac{4}{3}\pi}r$ and its surface area is $6\sqrt[3]{\frac{16}{9}\pi^2}r^2$. So the ratio of the surface area of the Sphere of Avarice to that of the Cube of Chaos is

$$\frac{4\pi r^2}{6\sqrt[3]{\frac{16}{9}\pi^2}r^2}=\sqrt[3]{\frac{\pi}{6}}$$

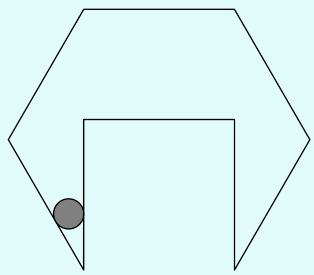
Now if the Cube of Ethan has surface area A_1, its side length is $\sqrt{\frac{A_1}{6}}$ and so its volume is $V_1 = \left(\frac{A_1}{6}\right)^{3/2}$. And if the Sphere of Splendor has surface area A_2, its radius is $\sqrt{\frac{A_2}{4\pi}}$ and so its volume is $V_2 = \frac{4}{3}\pi\left(\frac{A_2}{4\pi}\right)^{3/2}$. So our answer is

$$\frac{V_1}{V_2} = \frac{\left(\frac{A_1}{6}\right)^{3/2}}{\frac{4}{3}\pi\left(\frac{A_2}{4\pi}\right)^{3/2}} = \sqrt{\frac{\pi}{6}}\left(\frac{A_1}{A_2}\right)^{3/2} = \sqrt{\frac{\pi}{6}}\sqrt{\frac{\pi}{6}} = \boxed{\frac{\pi}{6}}$$

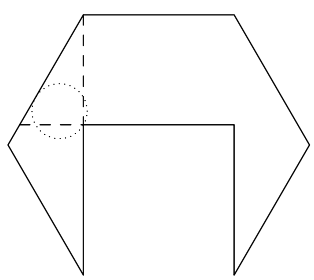
■

Proposed by Matthew Kroesche.

Team 7. Shown below is an eight-sided room in the shape of a regular hexagon of side length 12 feet, with a square of side length 12 feet removed from one of its sides. Two of the corners of the room are labeled "A" and "B", and a cannonball is wedged in the corner labeled "A". Nagito wants to roll the cannonball across the floor of the room and wedge it in the corner labeled "B" instead. What is the largest possible radius of the cannonball, in feet, such that it is possible for Nagito to do this? Express your answer in simplest radical form.



Solution. The choke point will be maneuvering the cannonball past either of the top two vertices of the square. If the sides of the square are extended to the opposite sides of the hexagon, we can realize this minimal distance as the altitude to the hypotenuse of a 30-60-90 right triangle.



Now this triangle has its longer leg of length $12\sqrt{3}-12$, and so the altitude to the hypotenuse has length half of this, which is $6\sqrt{3}-6$. The maximal radius of the cannonball is again half of this, which is $\boxed{3\sqrt{3}-3}$ feet. ■

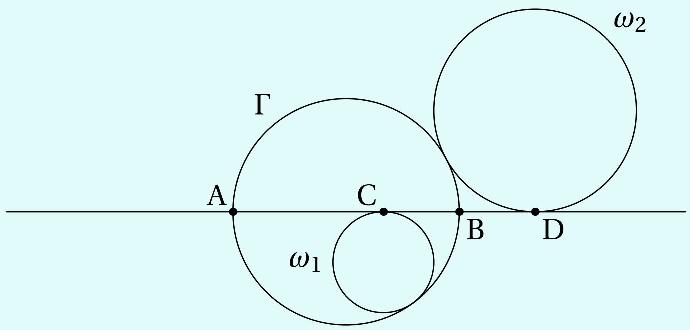
Proposed by Matthew Kroesche.
Remark: This problem is the worst.

Team 8. Luke is playing a game. Every minute, he scores one point with probability $\frac{1}{2}$, scores two points with probability $\frac{1}{4}$, or loses the game (and thus stops playing) with probability $\frac{1}{4}$. Luke wins the game (and thus stops playing) as soon as he has scored at least twelve points. What is the probability that Luke loses the game? Express your answer as a common fraction.

Solution. Let p_n denote the probability of Luke winning if he has n points left to score. Thus $p_0 = 1$, $p_1 = \frac{3}{4}$, and $p_{n+2} = \frac{1}{2}p_{n+1} + \frac{1}{4}p_n$. Running out this recursion for a few terms, we can see that $p_2 = \frac{5}{8}$, $p_3 = \frac{1}{2}$, and $p_4 = \frac{13}{32}$. We can solve the problem by either running out the recursion to p_{10}, or by noticing the Fibonacci sequence appearing in the numerator of p_n: in general we have $p_n = \frac{F_{n+3}}{2^{n+1}}$, which is satisfied by p_0 and p_1, and thus holds for all terms due to the recursion $F_{n+2} = F_{n+1} + F_n$. So $p_{10} = \frac{F_{13}}{2^{11}} = \frac{233}{2048}$. The probability that Luke loses the game is thus $1 - \frac{233}{2048} = \boxed{\frac{1815}{2048}}$. ∎

Proposed by Matthew Kroesche.

Team 9. Diameter AB of circle Γ has length 2. Points C, D lie on line AB such that C is inside Γ, D is outside it, and B is the midpoint of segment CD. Circle ω_1 is tangent to line AB at C and internally tangent to Γ. Circle ω_2 is tangent to line AB at D and externally tangent to Γ. Moreover, the sum of the areas of circles ω_1 and ω_2 equals the area of Γ. What is the positive difference between the radii of ω_1 and ω_2? Express your answer in simplest radical form.



Solution. Let $BC = BD = x$, and let O, P, Q be the centers of $\Gamma, \omega_1, \omega_2$ respectively. Let r_1, r_2 be the radii of ω_1, ω_2 respectively. By the Pythagorean Theorem on triangles OCP and ODQ, we have

$$(1-x)^2 + r_1^2 = (1-r_1)^2$$

$$(1+x)^2 + r_2^2 = (1+r_2)^2$$

and thus

$$r_1 = \frac{x(2-x)}{2}$$

$$r_2 = \frac{x(2+x)}{2}$$

We also know, since the areas sum to the area of Γ, that

$$\pi r_1^2 + \pi r_2^2 = \pi$$

$$\frac{\pi x^2(2-x)^2}{4} + \frac{\pi x^2(2+x)^2}{4} = \pi$$

$$x^2\left[(2-x)^2 + (2+x)^2\right] = 4$$

$$2x^4 + 8x^2 - 4 = 0$$

$$x^2 = \frac{-8 \pm \sqrt{64+32}}{4}$$

$$x^2 = \sqrt{6} - 2$$

Moreover, our answer is

$$r_2 - r_1 = \frac{x(2+x) - x(2-x)}{2} = x^2 = \boxed{\sqrt{6}-2}$$

■

Proposed by Matthew Kroesche.

Team 10. There are ten lilypads in a row in a pond. Each one is numbered with a different integer from 0 to 9 inclusive, such that the order of the numbers is chosen at random and all orderings are equally likely. Rankine the Toad starts at the leftmost lilypad, and repeatedly hops n lilypads to the right, where n is the number on the lilypad where he is currently located. If there are fewer than n lilypads to the right, Rankine lands in the pond and swims away. (So, for example, if the permutation of lilypads were 1403528796, the lilypads Rankine would visit would be, in order, 1, 4, 2, and 7, and after leaving 7 he would land in the pond and swim away.) What is the probability that Rankine eventually becomes stranded on the lilypad numbered 0? Express your answer as a common fraction.

Solution. We perform casework on the number of hops Rankine makes before first landing on the 0 lilypad. Note that after four hops without landing on 0, Rankine will have necessarily advanced at least $1+2+3+4=10$ lilypads to the right, and thus landed in the pond. Thus the number h of nonzero hops Rankine makes is either 0, 1, 2, or 3.
If $h=0$, then Rankine starts at zero; thus the leftmost lilypad is zero and the other nine lilypads can be arranged in any way, for a total of **9!** possible arrangements.
If $h=1$, then Rankine starts on some nonzero lilypad and advances directly to 0. There are 9 choices for the leftmost lilypad; for each of these choices, the position of the zero lilypad is uniquely determined, and the other eight lilypads can be arranged in any way, for a total of $\mathbf{9}\times\mathbf{8!}$ possible arrangements.
If $h=2$, then we consider where the zero lilypad is placed, determine the number of possible lengths for the two hops Rankine makes to reach it, and multiply by 7! since we do not care about the arrangement of the other seven lilypads. But this is equivalent to counting the number of ways to choose two distinct positive integers whose sum is less than 10. There are $7+6+5+4+4+3+2+1=32$ different ways to do this, so there are $\mathbf{32}\times\mathbf{7!}$ possible arrangements in this case.
Finally, for $h=3$, we count the number of ways to pick three distinct positive integers whose sum is less than 10, and multiply the result by 6!. We simply list all the different possible combinations of integers, irrespective of order (there aren't that many) and multiply that by $3! = 6$. We find the following seven combinations: $(1,2,3),(1,2,4),(1,2,5),(1,2,6),(1,3,4),(1,3,5),(2,3,4)$. Thus there are $7\times 6 = 42$ combinations, taking account of order, and so there are $\mathbf{42}\times\mathbf{6!}$ possible arrangements.

Now our answer is $\frac{9!+9\cdot 8!+32\cdot 7!+42\cdot 6!}{10!} = \frac{9\cdot 8+9\cdot 8+32+6}{10\cdot 9\cdot 8} = \frac{72+72+32+6}{720} = \frac{182}{720} = \boxed{\frac{91}{360}}$. ■

Proposed by Matthew Kroesche.
Remark: This problem has been on my desk since high school.

Countdown Problems

Remark: Note that these were all shuffled in a random order for the countdown round at the actual contest. All the same problems are here, just not in the same order.

Countdown 1. Two characters are standing on top of a high place, when they spot Dracula with a jetpack 50 meters away due south leisurely sipping coffee. They scream and start running away eastwards at 8 meters per second, while Dracula laughs. Fifteen seconds later, how far are they from Dracula?

Solution. We see that the characters travel $8 * 15 = 120$ meters, so using the Pythagorean theorem the answer is $\boxed{130}$ meters. ■

Proposed by Pierce Lai.

Countdown 2. What is $23 \times 26 \times 4 + 9$?

Solution. Solution: this is a 49 by 49 square. $49^2 = (50-1)^2 = \boxed{2401}$.

■

Proposed by Maeve Dever.

Countdown 3. Call an integer $a > 1$ *remarkable* if the union of the sets of the prime factors of a and $a+1$ is the set of the first k prime numbers, for some k. For example, 2020 is not remarkable, because the prime factors of 2020 and 2021 respectively are 2, 5, 101 and 43, 47, which are not the first 5 primes. However, 20 is remarkable, because the prime factors of 20 and 21 respectively are 2, 5 and 3, 7, which are the first 4 primes. What is the sum of the remarkable numbers less than 10?

Solution. Just start checking from the beginning. 2 works because of 2 and 3. 3 works because of 3 and 2. 5 works because of 5 and 2, 3. 8 works because of 2 and 3. 9 works because of 3 and 2, 5. The answer is $2+3+5+8+9 = \boxed{27}$. ■

Proposed by Arnav Adhikari.

Countdown 4. Alex is doing pulls in Boatknights. Suppose that every time he does a pull, he has a 2% chance of getting a 6-star operator, and each pull is independent. How many pulls does he need to do so that the expected number of 6-star operators he gets is 2?

Solution. Every pull gives a 0.02 expected number of 6-star operators, so he needs $2/0.02 = \boxed{100}$ pulls. ■

Proposed by Pierce Lai.

Countdown 5. Pierce is streaming Crossy Codes. Suppose he starts his stream on Thanksgiving of 2021, which is on a Thursday, and streams for a total of 365 days straight. How many Wednesdays will his stream cover?

Solution. 365 is 52 weeks and 1 day, so the answer is $\boxed{52}$. ■

Proposed by Pierce Lai.

Countdown 6. What is the value of $[(1-2)-(3-4)]-[(5-6)-(7-8)]$?

Solution. This is equal to $(-1--1)-(-1--1) = \boxed{0}$. ■

Proposed by Pierce Lai.

Countdown 7. Big Bob has a cube with surface area 600cm^2. What is the cube's volume in cubic centimeters?

Solution. Each face has area 100cm^3, so the cube has side length 10cm. Hence, the answer is $\boxed{1000}\text{cm}^3$. ■

Proposed by Pierce Lai.

Countdown 8. Badeline has half as much money as Monika, and twice as much as Kokichi. If, together, all three of them have \$21, how many dollars does Badeline have?

Solution. Monika counts as 4 Kokichis, so there are 7 Kokichi amounts total and so Kokichi has 3 dollars. Hence, Badeline has $\boxed{6}$ dollars, and Monika has 12 dollars. ■

Proposed by Pierce Lai.
Remark: No, this problem is the worst. - Josh

Countdown 9. Silence, Ifrit, and Saria are sharing a pizza that is cut into 8 slices. If Ifrit eats half of the pizza and Silence eats half of what remains, how many slices of pizza does Saria eat?

Solution. Saria eats $\frac{1}{4}$ of the pizza, or $\boxed{2}$ slices. ■

Proposed by Pierce Lai.

Countdown 10. If $\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{n}=1$, what is the value of n?

Solution. $\frac{1}{3}+\frac{1}{6}=\frac{1}{2}$, and $\frac{1}{2}+\frac{1}{4}+\frac{1}{5}=\frac{19}{20}$. Hence, n is $\boxed{20}$. ■

Proposed by Joshua Pate.

Countdown 11. The Taker is making chocolate pancakes for Lucy and Beel. Lucy wants 3 pancakes and Beel wants 5 pancakes. Their three pets, the Cerberus, also want 2 pancakes each. The Taker also wants 4 pancakes for himself. If each box of pancake mix can make 6 pancakes, then how many boxes of pancake mix do they need?

Solution. They want $3+5+2*3+4=18$ pancakes, so the answer is $\boxed{3}$ boxes of pancake mix. ■

Proposed by Pierce Lai.

Countdown 12. Josiah is playing Plants versus Zombies 2 with the Eclise mod. Suppose one Wall-nut has 3000 health. If a single zombie is eating this Wall-nut, the Wall-nut will last for 30 seconds before it is totally eaten. Now suppose Josiah uses plant food on a new Wall-nut, granting it a shell with an additional 16000 health. How long will this new Wall-nut last against a horde of 2 zombies?

Solution. Each zombie deals 100 damage per second to the nut, so the Wall-nut will last $19000/200 = \boxed{95}$ seconds. ■

Proposed by Pierce Lai.

Countdown 13. In CrineMath, the map is divided into chunks, which are regions that are 16 blocks long, 16 blocks wide, and 256 blocks tall. If 2^x is the number of blocks in one chunk, what is the value of x?

Solution. We see that the answer is $4+4+8 = \boxed{16}$. ■

Proposed by Pierce Lai.

Countdown 14. The winning marathon time at the 1896 Olympics was 2 hours, 58 minutes, 50 seconds. How many seconds is that?

Solution. Since this is 1 minute and 10 seconds short of three hours, the answer is $3*60*60-1*60-10 = 10800-70 = \boxed{10730}$ ■

Proposed by Joshua Pate.

Countdown 15. Alex is playing Boatknights. Every 15 seconds, he hears Krose say "Ko~Ko~Da~Yo~", which annoys him a very minuscule amount. Suppose Alex can withstand hearing "Ko~Ko~Da~Yo~" one thousand times before he cracks. If he starts continuously playing Boatknights at 9:00 AM, to the nearest minute at what time will he crack?

Solution. It will take about 15000 seconds, or 4 hours and 10 minutes. Hence, Alex will crack at $\boxed{1:10PM}$. ■

Proposed by Pierce Lai.

Countdown 16. Matthew is playing Crossy Codes. Every 9 seconds, an orange moth will shoot a fireball at him. Every 10 seconds, a blue moth will shoot a laser at him. Every 12 seconds, a jellyfish will launch a superheated water bubble towards him. Suppose that all three things occur at the same time. How long in minutes is the next time all three things occur concurrently?

Solution. We see that the lcm of 9, 10, and 12 is 180, so the answer is $\boxed{3}$ minutes. ■

Proposed by Pierce Lai.

Countdown 17. Jyu is eating almonds. One handful of almonds is 15 almonds, and one mouthful of almonds is 24 almonds. If a bag of almonds is both a whole number of handfuls of almonds and also a whole number of mouthfuls of almonds, then what is the smallest possible (positive) number of almonds in one bag of almonds?

Solution. The answer is just the lcm of 15 and 24, which is $\boxed{120}$ almonds. ■

Proposed by Pierce Lai.

Countdown 18. Kat has three regular unit squares. She glues these squares together to form one large shape, and then calculates the sum of the interior angles of this shape in degrees. What is the least possible value of this sum?

Solution. The answer is $\boxed{360}$ degrees (a rectangle). ■

Proposed by Pierce Lai.

Countdown 19. What is $3 \times (3+3) - 3^3/3$?

Solution. The answer is $3 \times 6 - 27/3 = 18 - 9 = \boxed{9}$. ■

Proposed by Pierce Lai.

Countdown 20. A number of people are writing problems for a certain math competition. Suppose that they work for 24 hours per day, and that every hour they churn out 15 caffeinated math problems. Suppose further that every page of this math competition contains 1200 problems, and that the competition contains x pages in total. If they must work for 10 days straight to write enough problems for the competition, how many pages long is the competition?

Solution. We see that $x * 1200/24/15 = x * 10/3 = 10$, so $x = \boxed{3}$ pages. ■

Proposed by Pierce Lai.

Countdown 21. Suppose Pierce is baking cookies. He has unfortunately run out of eggs, but he does have some liquid egg substitute. Suppose 65 milliliters of egg substitute can do the work of one medium egg and Pierce has 5 liters of the stuff. If a batch of 48 cookies requires one egg, what is the maximum whole number of dozens of cookies Pierce can make with his egg substitute?

Solution. $5000/65 = 1000/13 \approx 76.9$. Hence, as 48 is 4 dozen, we can make $76 * 4 + 3 = \boxed{307}$ dozen cookies. ■

Proposed by Pierce Lai.

Countdown 22. What is $251^2 - 1$?

Solution. The answer is $(251-1)(251+1) = (250)(252) = (1000)(63) = \boxed{63000}$ ■

Proposed by Joshua Pate.

Countdown 23. What is $29^2 + 96^2$?

Solution. Since $29^2 = 28^2 + 2 * 28 + 1$, and $28^2 + 96^2 = (4^2)(7^2 + 24^2) = (4^2)(25^2) = 10000$, $29^2 + 96^2 = 1 + 2 * 28 + (28^2 + 96^2) = 1 + 56 + 10000 = \boxed{10057}$. ■

Proposed by Joshua Pate.

Countdown 24. In Sassafras City, there are seven sheiks, precisely six of whom are shrieking sheiks. Each sheik has seven sheep, precisely six of whom are sick. Each sheep sells seven seashells, precisely six of which are sold by the seashore. How many seashells are sold by sick sheep by the seashore in Sassafras City?

Solution. There are a total of seven sheiks, each of whom has six sick sheep, and each sick sheep sells six seashells by the seashore. Thus we answer $7 \cdot 6 \cdot 6 = \boxed{252}$. ■

Proposed by Matthew Kroesche.
Remark: It had to be done.

Countdown 25. The Bad Place is delivering a shipment of Hawaiian pizzas. The shipment consists of eight crates, each of which contains ten pizza boxes. If each pizza box contains a single large twelve-slice pizza, and each person eats three slices of pizza, how many people can the shipment of pizzas feed?

Solution. In total, there are $8 \times 10 \times 12 = 960$ slices of pizza, so the shipment can feed $\frac{960}{3} = \boxed{320}$ people. (Almost enough for the whole neighborhood...) ■

Proposed by Matthew Kroesche.

Countdown 26. In BTD6, a MOAB has 200 health, a BFB has 700 health, a ZOMG has 4000 heath, and a BAD has 20000 health. If a selection of enemies from the four above has a total of 35000 health, what is the least total number of MOABs and BFBs in the selection?

Solution. The minimal number is clearly given by 3000 health's worth, 4 BFBs and an MOAB, so the answer is $\boxed{5}$. (The other 32000 could be obtained by a BAD and 3 ZOMGs, or 8 ZOMGs. ■

Proposed by Pierce Lai.

Countdown 27. Matthew has an infinite chocolate fondue fountain, except it's not actually chocolate; it's clam chowder which has the same consistency and color as chocolate. Suppose the fountain generates 3 liters of the brownish liquid every minute. If a serving of chowder is 250 ml and a person consumes 2 servings of chowder per meal, and eats 3 meals per day, how many people can the fountain serve?

Solution. There are 1440 minutes in a day, and each person eats 1.5 liters of chowder per day, or half a minute's worth. Hence, the answer is $\boxed{2880}$. ■

Proposed by Pierce Lai.

Countdown 28. Scaggy and Shoob have a function, $f(x) = x^4 - 4x^3 + 6x^2 - 4x + 2$. What is the value of $f(-2) + f(-1) + f(0) + f(1) + f(2)$?

Solution. We see that $f(x) = (x-1)^4 + 1$. Hence, the answer is $82 + 17 + 2 + 1 + 2 = \boxed{104}$. ■

Proposed by Pierce Lai.

Countdown 29. Pierce has a lot of peppermints. Let P be the number of peppermints he has. If P has a remainder of 1 when divided by 3, a remainder of 3 when divided by 5, and a remainder of 5 when divided by 7, what is the least possible value of P?

Solution. Notice that all the remainders are equivalent to -2 modulo what they are divided by. Hence, the answer is $3 * 5 * 7 - 2 = \boxed{103}$ peppermints. ■

Proposed by Pierce Lai.

Countdown 30. Pierce and Alex are playing the game CrineMath. Every time they earn a stash of loot, there is a 20% chance that Pierce will accidentally blow it up. How many stashes of loot must Pierce and Alex earn to ensure that the expected number of unexploded stashes is at least 12?

Solution. Taking $12/0.8 = \boxed{15}$ farmings. ■

Proposed by Pierce Lai.

Countdown 31. John is busy turning a pool of water into cranberry juice. Except his friend, Jones doesn't like cranberry juice, so Jones is turning some of it back into water. Suppose John working alone can turn a pool of water into cranberry juice in 15 hours, and Jones working alone can turn a pool of cranberry juice into water in 20 hours. If the pool starts out $1/3$ cranberry juice, how many more hours will it take for John to finish turning the pool into cranberry juice?

Solution. $\frac{1}{15} - \frac{1}{20} = \frac{1}{60}$, so John would take 60 hours from a fresh pool. Since the pool is already $1/3$ wine, the answer is $\frac{2}{3}60 = \boxed{40}$ hours. ■

Proposed by Pierce Lai.

Countdown 32. If $16^{(16^{16})} = (16^{16})^n$ and $n = 2^m$, what is the value of m?

Solution. The left side is equal to $2^{2^{66}}$ and the right side is equal to $2^{2^6 n}$, so $n = 2^{60}$. Hence, $m = \boxed{60}$. ■

Proposed by Pierce Lai.

Countdown 33. Josh claims to have written N questions for the Awfully Mediocre Convention of Peculiar Mathematics (AMCPM) but hasn't added any of them to the contest yet. If Josh adds three questions to the shortlist every day, starting today, he would add the last three of his N questions to the shortlist on the day before the deadline. However, if Josh waits another week without doing anything and starts adding questions to the shortlist a week from today, he would need to add four questions to the shortlist every day to finish before the deadline (and in this case, he would add the last four of his questions on the day before the deadline). Given this information, what is the value of N?

Solution. The number of days until the deadline is $\frac{N}{3}$, and also a week from today, the number of days until the deadline will be $\frac{N}{4}$. Thus, $\frac{N}{3} - \frac{N}{4} = 7$ and so $N = 4N - 3N = \boxed{84}$. ■

Proposed by Matthew Kroesche.

Countdown 34. Isabella has baked a strawberry pie. She picks a random integer from 1 to 256 inclusive. If it is divisible by 3, Chris sets the pie on fire. If it is divisible by 4, Chris skewers the pie. If it is divisible by 5, Chris throws the pie at Rich. What is the probability at least two of these things will happen? Express your answer as a common fraction.

Solution. There are 21 numbers divisible by 12, 17 by 15, 12 by 20, and 4 by 60, so there are $21+17+12-4*2=42$ valid numbers. Hence, the answer is $\frac{42}{256}=\boxed{\frac{21}{128}}$. ■

Proposed by Pierce Lai.

Countdown 35. Alex is eating a bar of tasty Hershey's king-sized buttersquash pea and nut 99% dark chocolate. Suppose that if the bar's length were increased by 6 units and its width were increased by 2 units, the size of the bar would be doubled. If the length of the bar is twice its width, what is the current length of the bar?

Solution. We have that $4w^2=(w+2)(2w+6)=2w^2+10w+12$. Hence, $w^2-5w-6=0$, so $w=6$. Hence, the length of the bar is $\boxed{12}$ units. ■

Proposed by Pierce Lai.

Countdown 36. Jx is eating chicken pot pie. The pot pie is a perfect cylinder, with radius 10 feet and thickness 3 feet. The pie also has crunchy savory crust on the bottom and sides. In square feet, what is the amount of crust on the pot pie? Express your answer in terms of π.

Solution. The bottom has area 100π and the sides have area 60π, so the answer is $\boxed{160\pi}$ square feet of crust. ■

Proposed by Pierce Lai.

Countdown 37. Suppose a recipe requires precisely 46.5 cups of sugar, but you only have a 17 cup measuring cup, a 7 cup measuring cup, and a 1.5 cup measuring cup. What is the least number of times you have to use the cups?

Solution. Notice that you can use the 17 cup 3 times for 51 cups of sugar, and then remove 1.5 cups 3 times, for a total of $\boxed{6}$ uses. Alternatively, one can use the 17 cup once, the 7 cup 4 times, and then the 1.5 cup once. ■

Proposed by Pierce Lai.

Countdown 38. Bobby and Joeby are playing a game with two regular, 6-sided dice. On their turn, they roll the dice, and if the sum of the two rolls is a prime number, they win. Otherwise their turn ends and the turn goes to the other player. If Bobby goes first, what is the probability that he wins the game? Express your answer as a common fraction.

Solution. Every turn has a $\frac{1+2+4+6+2}{36}=\frac{5}{12}$ chance of resulting in a win. Thus the answer is $\frac{5}{12}(1+(\frac{7}{12})^2+(\frac{7}{12})^4+\dots)=\frac{5}{12}\frac{1}{1-\frac{49}{144}}=\frac{5}{12}\frac{144}{95}=\boxed{\frac{12}{19}}$. ■

Proposed by Pierce Lai.

Countdown 39. Honey, Nut, and Cheerio are collecting honey nut cheerios. Suppose Honey has 2 fewer honey nut cheerios than twice Nut's collection, Nut has 3 fewer honey nut cheerios than thrice Cheerio's collection, and Cheerio has 4 fewer honey nut cheerios than four times Honey's collection. How many cheerios do they have in total? Express your answer as a common fraction.

Solution. We see that Nut has 12 times Honey's collection minus 15, so Honey has 24 times Honey's collection minus 32. Hence, Honey must have $\frac{32}{23}$ cheerios, Nut has $\frac{39}{23}$ cheerios, and Cheerio has $\frac{36}{23}$ cheerios. Hence, altogether they have $\frac{32+39+36}{23} = \boxed{\frac{107}{23}}$ cheerios. ∎

Proposed by Pierce Lai.

Countdown 40. Jyu is teaching a class on high-level tetris. Suppose he assigns a test, and the average test score for the entire class is a 78. Suppose that 3/5 of the class studied for the test, and those who did had an overall average of a 84 on the test. How many points better, on average, did those who studied do than those who did not?

Solution. By observation, those who did not study averaged $78 - 9 = 69$ (nice), so the answer is $84 - 69 = \boxed{15}$ points. ∎

Proposed by Pierce Lai.

Countdown 41. Matthew and his alternate form, Matthew-chan, are playing a variant of tic tac toe. Matthew selects 4 squares out of 9 at random on the board, and he wins if 3 of them are in a line. Otherwise, Matthew-chan wins. What is the probability that Matthew wins? Express your answer as a common fraction.

Solution. There are $\binom{9}{4} = 126$ possible choices. There are 8 ways to pick a row, column, or diagonal, and there are 6 choices for the last square, so 48 total choices. Hence, the answer is $\frac{48}{126} = \boxed{\frac{8}{21}}$ ∎

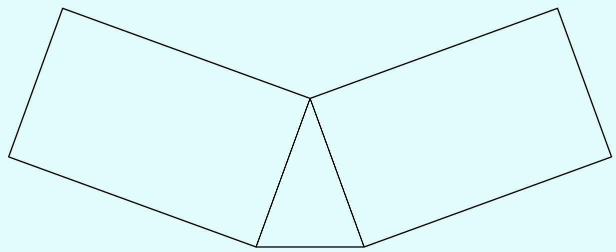
Proposed by Pierce Lai.

Countdown 42. A circular target consists of a small circle of radius 1, and then three concentric rings around it, each with 1 greater radius. Suppose the outermost ring has score 1, the second outermost ring has score 2, the second innermost ring has score 3, and the circle has score 4. If a monkey throws a dart at the target and it lands uniformly randomly across the target, what is the expected value of the score of the sector it lands on?

Solution. Notice that this is equal to adding 1 value per unit space per circle (as every sector has 1 more score than the previous sector). Hence, the expected score is $(\pi + 4\pi + 9\pi + 16\pi)/16\pi = \boxed{\frac{15}{8}}$. Express your answer as a common fraction. ∎

Proposed by Pierce Lai.

Countdown 43. FancyShirtLand is trying to plan out their new line of polo shirts, but unfortunately they have no mathematicians on staff to optimize the collar. Specifically, the collar consists of two congruent 3×5 rectangles that touch at a single vertex such that the two sides touching the acute angle formed between the rectangles are congruent. Then, line segments are drawn between the lowest vertices as well as the highest vertices (to emphasize the front and back of the shirt). FancyShirtLand wants to maximize the height of the collar i.e. the distance between the low and high line segments. What is the maximum possible height? Express your answer in simplest radical form.



Solution 1. Consider the isosceles trapezoid with legs as the diagonals of the rectangles and bases as the two line segments. The legs have length $\sqrt{34}$, so the Pythagorean theorem implies that the height of the trapezoid must be at most $\boxed{\sqrt{34}}$. Indeed, a working construction is where the diagonals are actually perpendicular to the line segments. ■

Solution 2. Let A be the shared vertex, B_1 and B_2 be the lowest vertices from left to right, C_1 and C_2 be the highest vertices from left to right, and H_B, H_C be the projections, respectively, of A onto B_1B_2 and C_1C_2. We wish to maximize $AH_B + AH_C$. Since the collar is symmetric, we actually don't need to deal with B_2 and C_2.

Note that $\angle B_1AB_2 + \angle C_1AC_2 = 180^\circ \implies \angle H_BAB_1 + \angle H_CAC_1 = 90^\circ$, so $\triangle AB_1H_B \sim C_1AH_C$. If we let $AH_B = 3x$, then $B_1H_B = 3\sqrt{1-x^2}$. Using similarity ratios, we compute $C_1H_C = 5x$ and $AH_C = 5\sqrt{1-x^2}$. Thus we need to maximize $3x + 5\sqrt{1-x^2}$. By Cauchy-Schwarz,

$$(9+25)\left(x^2 + \sqrt{1-x^2}^2\right) \geq \left(3x + 5\sqrt{1-x^2}\right)^2 \implies 3x + 5\sqrt{1-x^2} \leq \boxed{\sqrt{34}}.$$

Equality holds when $\frac{3}{x} = \frac{5}{\sqrt{1-x^2}} \implies x = \frac{3}{\sqrt{34}} \implies AH_B = \frac{9}{\sqrt{34}}, AH_C = \frac{25}{\sqrt{34}}$. ■

Proposed by Arnav Adhikari.

Countdown 44. Suppose x and y are positive integers such that they satisfy $2^x = y^2$ and y/x is a positive prime. What is the value of $x + y$?

Solution. We see that y must be a power of 2, so we have that $y = 2^a$ for some a and $x = 2a$. $y/x = 2^{a-1}/a$, so a must also be a power of 2. In addition, clearly the only prime $2^{a-1}/a$ could be is 2. Hence, the smallest valid value of a is 4, giving $y = 16$ and $x = 8$, so the answer is $x + y = \boxed{24}$. ■

Proposed by Pierce Lai.

Countdown 45. Hm has a fair coin. Suppose that he picks a face of the coin, and independently and randomly picks either 0 or 1 (each with 50% probability) and paints that number of dots on that face. Hm does the same thing for the other face, so now both faces have either 0 or 1 dots. Hm then flips the coin twice. What is the probability that the number of dots on the top face of coin sums to 1 across the two flips? Express your answer as a common fraction.

Solution. The coin can have either 2 faces with 0 dots (1/4 chance), 1 face with 0 dots and 1 with 1 (1/2 chance), or 2 faces with 1 dot each. The only way for the sum to be 1 is the second case, and assuming the second case that occurs with 1/2 chance. Hence, the answer is $(1/2)(1/2) = \boxed{1/4}$. ■

Proposed by Pierce Lai.

Countdown 46. Ethan is doing an experiment. He is putting sawdust in Rice Krispies, and seeing how long it takes for someone to notice. Suppose a modified Rice Krispie has a mass of 100 kg, of which 30% is sawdust. Ethan adds sawdust to the Rice Krispie so now the % of sawdust is doubled to 60%. How much sawdust in kg did he add?

Solution. The amount of non-sawdust Rice Krispie is 70 kg, which is 40% of the new Rice Krispie. Hence, the new Rice Krispie is 175 kg, so Ethan added $\boxed{75}$ kg of sawdust. ■

Proposed by Pierce Lai.

Countdown 47. If p is a positive real number satisfying $p(2-p) = (1-2p)(1+2p)$, what is the value of p? Express your answer as a common fraction.

Solution. Rearranging terms gives $2p-p^2+4p^2-1 = 3p^2+2p-1 = (3p-1)(p+1) = 0$. As p is a positive real number, p can't be -1, so p must equal $\boxed{\frac{1}{3}}$. ■

Proposed by Pierce Lai.

Countdown 48. What is $\frac{2^{(2^{(2^2)})}-1}{((2^2)^2)^2-1}$?

Solution. The numerator equals $2^{16}-1$ and the denominator 2^8-1, so the answer is $2^8+1 = \boxed{257}$. ■

Proposed by Joshua Pate.

Countdown 49. Suppose two positive numbers x, y satisfy $\frac{1}{x}+\frac{1}{y} = 1$. If $x = \sqrt{2}$, what is y? Express your answer in simplest radical form.

Solution. This gives $y = \frac{1}{1-\frac{1}{\sqrt{2}}} = \frac{\sqrt{2}}{\sqrt{2}-1} = \boxed{2+\sqrt{2}}$. ■

Proposed by Pierce Lai.

Countdown 50. Alphys, Beta-carotene, and Catarina pick three positive numbers a, b, and c respectively. The values $\gcd(a,b)$, $\gcd(a,c)$, and $\gcd(b,c)$ are all distinct. What is the smallest possible value of $a+b+c$?

Solution. Notice that the smallest values the gcds can be are 1, 2, and 3. Using this gives us $2+3+6 = 11$.

We claim that this is the answer. To show this, first notice that a, b, and c must all be distinct (otherwise two gcds would be the same), and must all be at least 2 (otherwise at least two gcds would be 1). Thus, the smallest possible value is $2+3+4 = 9$. For 9, we see that $2+3+4$ does not work, and for 10 the only solution $2+3+5$ also does not work. Hence, the answer is $\boxed{11}$. ■

Proposed by Pierce Lai.

Countdown 51. What is $\frac{8^8+1}{4^4+1}$?

Solution. Since $4^4 = 2^8$ and $8^8 = 2^24 = (2^8)^3$, $\frac{8^8+1}{4^4+1} = (2^8)^2 - 2^8 + 1 = 2^16 - 2^8 + 1 = 65536 - 256 + 1 = \boxed{65281}$. ■

Proposed by Joshua Pate.

Countdown 52. Lorp has two regular unit squares. She glues these squares together to form one large shape, and then calculates the sum of the interior angles of this shape in degrees. What is the maximum possible value of this sum?

Solution. The answer is $\boxed{1080}$ degrees (a peanut). ■

Proposed by Pierce Lai.

Countdown 53. Eeng the walmart Avatar is on a quest to collect 7 dragonballs, in order to defeat the fire lord Flame-o Hotman before the world hits Armageddon and is drowned in 40 days of apple pie filling. Suppose that Eeng takes 3 days to collect the first dragonball, and that collecting each dragonball after takes twice the number of days as the previous dragonball. How many days will Eeng take to collect all 7 dragonballs?

Solution. We see that the answer is $3(1+2+2^2+2^3+2^4+2^5+2^6) = 3(127) = \boxed{381}$ days. ■

Proposed by Pierce Lai.
Remark: What even. -Matthew

Countdown 54. Pierce is doing Duolingo. His goal is to get a 10000 day streak on Duolingo. Suppose he starts doing Duolingo on a Wednesday, which is day 1 of his streak. If he doesn't miss a single day, on what day of the week will he reach 10000 days?

Solution. He will reach a 10000 day streak 9999 days after Wednesday, which is congruent to 3 mod 7. Hence, the answer is $\boxed{Saturday}$. ■

Proposed by Pierce Lai.

Countdown 55. Two teams are playing volleyball. Serves succeed with 100% probability, and when either team receives a ball, they have a 50% chance of returning it to the other team. When the ball is served (which makes the ball pass to the other team), what is the expected number of times the ball is passed between the teams (including the initial serve)?

Solution. We see that this is just equal to $1 + \frac{1}{2} + \frac{1}{4} \cdots = \boxed{2}$. ■

Proposed by Pierce Lai.

Countdown 56. Lea is eating a very large sandwich. Eating alone, she can eat the entire sandwich in 10 minutes. However, if Lea and her friend, Emilie, work together to eat the sandwich, it only takes them 6 minutes to eat the sandwich. Supposing the two eat at constant rates, how long in minutes would it take for Emilie to eat the sandwich by herself?

Solution. Lea eats at a rate of $\frac{1}{10}$ of the sandwich per minute, while Lea and Emilie eat at a combined rate of $\frac{1}{6}$ of the sandwich per minute. Hence, Emilie alone eats at a rate of $\frac{1}{6} - \frac{1}{10} = \frac{4}{60} = \frac{1}{15}$ of the sandwich per minute, so the answer is $\boxed{15}$ minutes. ■

Proposed by Pierce Lai.

Countdown 57. The potato vendor Closure has a shipment of potatoes. Each potato has either 6 eyes or 17 eyes. Given that her potatoes have 100 eyes total, how many potatoes does she have?

Solution. Notice that $100 \equiv 4 \pmod 6$ and $17 \equiv 5 \pmod 6$. Hence, she has 2 potatoes with 17 eyes and 11 potatoes with 6 eyes, for a total of $\boxed{13}$ potatoes. ■

Proposed by Pierce Lai.

Countdown 58. Matthew keeps his socks in a jar. He has 4 green socks, 4 blue socks, and 4 red socks. If he picks 2 socks out from his jar randomly without replacement, what is the probability they are the same color? Express your answer as a common fraction.

Solution. The answer is $\frac{3\binom{4}{2}}{\binom{12}{2}} = \frac{18}{66} = \boxed{\frac{3}{11}}$. ■

Proposed by Pierce Lai.

Countdown 59. Lumpkin and Bumpkin harvested some pumpkins. If Lumpkin had harvested seven more pumpkins, he would have harvested twice as many pumpkins as Bumpkin. If instead Bumpkin had harvested sixteen more pumpkins, he would have harvested twice as many pumpkins as Lumpkin. How many pumpkins did Lumpkin and Bumpkin harvest together?

Solution. Suppose Lumpkin and Bumpkin harvested x and y pumpkins, respectively. Then we have

$$x + 7 = 2y$$

$$y + 16 = 2x$$

Adding these together gives

$$x + y + 23 = 2(x + y)$$

and so $x + y = \boxed{23}$. (We can also solve the system explicitly to get $x = 13$ and $y = 10$.) ■

Proposed by Matthew Kroesche.

Countdown 60. What integer is closest to $\sqrt{1015059}$?

Solution. Since $1007.5^2 = (1000 + 7.5)^2 = 1000^2 + 2 * 7.5 * 1000 + 7.5^2 = 1000000 + 15000 + 56.25 = 1015056.25$, $\sqrt{1015059} > 1007.5$, so it is closest to $\boxed{1008}$. ■

Proposed by Joshua Pate.

Countdown 61. What is the sum of the prime factors of 1729?

Solution 1. Testing low level primes, we see that $1729 = 7 * 247$. Either by continued testing, or observation that $247 = 16^2 - 3^2$, we have that $1729 = 7 * 13 * 19$, so the answer is $7 + 13 + 19 = \boxed{39}$. ■

Solution 1. Noticing that $1729 = 10^3 + 9^3 = 19(10^2 - 9 * 10 * 9^2) = 19 * 91 = 19 * 7 * 13$, we see that the answer is $\boxed{39}$. ■

Proposed by Joshua Pate.

Countdown 62. Pierce loves sugar. He has a large rectangular prism of sugar, consisting of 1400 individual sugar cubes (glued together with, you guessed it, more sugar!), such that each side measures an integer number of sugar cubes. Suppose Pierce dunks his block of sugar into a vat of cherry-flavored coating so that the entire outside of the block is colored red, and then takes the block out and carves it up into the individual sugar cubes. Let M be the number of sugar cubes that have at least one face colored red by the cherry-flavored coating. What is the minimum value of M?

Solution. Note that $1400 = 2^3 5^2 7$. To minimize M, we want to essentially make the rectangular prism as cubical as possible (each side should be around 11-12 sugar cubes). The solution is pretty trivially $10 * 10 * 14$ (7 by 200 is quite unoptimal). Hence, the answer is $1400 - 8 * 8 * 12 = \boxed{632}$ sugar cubes. ■

Proposed by Pierce Lai.

Countdown 63. Unpyrus, Pans, and Sadyne are standing at the vertices of an equilateral triangle of side length 10 meters. Pans then struts dead-pannedly a distance of x meters while Sadyne and Unpyrus stay still, so that the three now form three vertices of a square. What is the least value of x, in meters? Express your answer in simplest radical form.

Solution. We see that the square is the one with Sadyne and Unpyrus as a diagonal, which gives us $x = \boxed{5\sqrt{3} - 5}$ meters. ■

Proposed by Pierce Lai.

Countdown 64. Every time a comet passes near the Earth, there is a one in three chance that Princess Peach will be kidnapped, a one in four chance that the Fire Nation will try to take over the world, and a one in five chance that a falling meteor will smite the town of Itomori. When Halley's comet appears in 2061, what is the probability that at least two of these three events happen (assuming they are independent)? Express your answer as a common fraction.

Solution. The answer is

$$\frac{1}{3} \cdot \frac{1}{4} \cdot \frac{4}{5} + \frac{1}{3} \cdot \frac{3}{4} \cdot \frac{1}{5} + \frac{2}{3} \cdot \frac{1}{4} \cdot \frac{1}{5} + \frac{1}{3} \cdot \frac{1}{4} \cdot \frac{1}{5} = \frac{4+3+2+1}{3 \cdot 4 \cdot 5} = \frac{10}{60} = \boxed{\frac{1}{6}}$$

■

Proposed by Matthew Kroesche.

Countdown 65. Sam likes potatoes. Every day at dinnertime, he cooks potatoes either by boiling them, mashing them, or sticking them in a stew. (Each of these three ways is chosen randomly, equally likely, and independent of what he chooses on other days.) What is the probability that Sam serves mashed potatoes at least twice this week? Express your answer as a common fraction.

Solution. The number of possible outcomes for this week is 3^7, and the number of ways for Sam to serve mashed potatoes at most once this week is $2^7+7\cdot 2^6 = 2^6\cdot 3^2$. Thus the probability that Sam serves mashed potatoes at most once is $\frac{2^6\cdot 3^2}{3^7} = \frac{2^6}{3^5} = \frac{64}{243}$, so we answer $1-\frac{64}{243} = \boxed{\dfrac{179}{243}}$. ■

Proposed by Matthew Kroesche.

Countdown 66. Matthew sets his alarm to go off at 8:00 AM. However, he has a bad habit of hitting the snooze button as soon as the alarm goes off, which stops it, but causes it to go off again exactly nine minutes later. To mitigate this problem, Matthew has also set separate alarms to go off at 8:05, 8:10, and every 5-minute interval after 8:00. He then proceeds to hit the snooze button on every one of these alarms, causing each one to keep going off forever every nine minutes. What is the largest positive integer m such that no alarms on Matthew's phone go off exactly m minutes after 8:00 AM?

Solution. By the Chicken McNugget theorem, the answer is $9\cdot 5-9-5 = \boxed{31}$. ■

Proposed by Matthew Kroesche.
Remark: This problem is less fictional than I'd like to admit.

Countdown 67. Josh is binge watching an extremely long-running TV show. He starts watching on December 30, 2020, and watches five episodes every day until January 15, 2022, when he gets to the twist ending in the final episode that retroactively ruins the plot of the entire series. If every season of the show is divided into six chapters, and each chapter has six episodes, then how many seasons long is the show?

Solution. In total, Josh watched the show for $2+365+15 = 382$ days. Every 36 days, he finished exactly 5 seasons, so after 360 days, he had finished 50 seasons. In the remaining 22 days, he watched between 106 and 110 episodes (since $22\times 5 = 110$ but he may not have watched a whole five episodes on the last day). Thus he watched $\frac{108}{36} = 3$ seasons in those 22 days, so in total there must be $\boxed{53}$ seasons. ■

Proposed by Matthew Kroesche.
Remark: bruh

Countdown 68. In the spring interhigh volleyball tournament, Tsubakiduba High School and Shiratirazira High School play each other in a best-of-three match. (Thus, they play three sets, and whichever team wins two or more sets wins the match.) If Shiratirazira wins each set with probability 60% (independently of all other sets), what is the percentage probability that they win the match? Express your answer as a decimal to the nearest tenth.

Solution. Let $p = \frac{3}{5}$ be the probability that Shiratirazira wins a set. Then the probability that they win all three is $p^3 = \frac{27}{125}$. The probability that they win two (and Tsubakiduba wins the remaining one) is $3p^2(1-p) = \frac{54}{125}$. Thus, the probability that they win at least two is $\frac{27}{125}+\frac{54}{125} = \frac{81}{125}$, which is $\boxed{64.8\%}$. ■

Proposed by Matthew Kroesche.
Remark: I regret nothing.

Countdown 69. Find the average of the reciprocals of the positive integer divisors of 120. Express your answer as a common fraction.

Solution. We factor $120 = 2^3 \cdot 3 \cdot 5$ and see that it has 16 divisors. Moreover, their sum is $(2^3 + 2^2 + 2 + 1)(3 + 1)(5 + 1) = 15 \cdot 4 \cdot 6 = 360$, and so the sum of their reciprocals is just $\frac{360}{120} = 3$. Thus the answer is $\boxed{\dfrac{3}{16}}$. ■

Proposed by Joshua Pate.

Challenge Problems

Challenge 1. What is the value of 11_4 divided by 3_4 in base 4? Express your answer as a repeated decimal.

Solution. In base 10, $11_4 = 5$ and $5/3 = 1\frac{2}{3}$. Hence the answer is $\boxed{1.\bar{2}}$. ■

Proposed by Pierce Lai.

Challenge 2. Pierce has some popcorn which he claims is still edible. If today is a Thursday and its best by date was 400 years ago to the day, on what day of the week was its best by date?

Solution. 400 years covers $100-4+1=97$ leap years. Hence, $400*365+97 \equiv 1*1+6 \equiv 0 \pmod 7$, so the answer is $\boxed{Thursday}$. ■

Proposed by Pierce Lai.
Remark: Definitely not based on any real life events.

Challenge 3. Pierce has a box of size 5 units by 5 units by 5 units, and he is placing depleted uranium rods into the box. These rods are rectangular prisms of size 2 units by 2 units by 3 units. What is the maximum number of rods he can fit in the box?

Solution. I think the answer is 8. Not sure though. ■

Proposed by Pierce Lai.

Challenge 4. What is the largest integer with distinct digits (in base 10) that is divisible by 7?

Solution. We see that $98765 \equiv 2 \pmod 7$, so 987654321 is not divisible by 7. By guess and check, the largest integer is $\boxed{9876543201}$. ■

Proposed by Pierce Lai.

Challenge 5. There are a total of N teams applying to compete at the the Horribly MisManaged Tournament (HMMT). However, the organizers randomly accept only 150 of these N teams to actually compete. If Matthew applies three teams to compete, what is the least value of N such that the probability that none of Matthew's teams are accepted is greater than the probability that exactly one of them is accepted?

Solution. The probability that none are accepted is

$$\frac{\binom{N-3}{150}}{\binom{N}{150}} = \frac{\frac{(N-3)(N-4)\cdots(N-152)}{150!}}{\frac{N(N-1)\cdots(N-149)}{150!}} = \frac{(N-150)(N-151)(N-152)}{N(N-1)(N-2)}$$

And the probability that exactly one of them is accepted is

$$\frac{\binom{N-3}{149}}{\binom{N}{150}} = \frac{\frac{(N-3)(N-4)\cdots(N-149)(N-150)(N-151)}{149!}}{\frac{N(N-1)\cdots(N-149)}{150!}} = \frac{150(N-150)(N-151)}{N(N-1)(N-2)}$$

So we want to solve

$$(N-150)(N-151)(N-152) > 150(N-150)(N-151)$$

Thus we have $N-152 > 150$, so $N > 302$. Thus we answer $\boxed{303}$. ■

Proposed by Matthew Kroesche.

Challenge 6. Mario is trying to open eight treasure chests. However, he has to open them in a specific and predetermined order, which he does not know. If he opens a chest out of order, all the chests slam shut and he gets zapped, and then he has to start over. He has a perfect memory, and he will not ever open a chest he knows will zap him. But other than that, he chooses randomly. What is the expected number of times he gets zapped before he opens all eight chests in the correct order?

Solution. The expected number of times he gets zapped until he figures out the correct first chest to open is $\frac{7+6+5+4+3+2+1+0}{8} = \frac{28}{8} = \frac{7}{2}$. In general, when there are n chests whose ordering he hasn't figured out yet, he will get zapped $\frac{0+1+\cdots+(n-1)}{n} = \frac{n-1}{2}$ times on average before figuring it out. Thus the answer is

$$\frac{7}{2} + \frac{6}{2} + \cdots + \frac{1}{2} = \frac{28}{2} = \boxed{14}$$

■

Proposed by Matthew Kroesche.

Challenge 7. An ant is on a vertex of a dodecahedron. At the end of each minute, the ant moves from the vertex it is on to a neighboring vertex. What is the probability that the ant is on the vertex opposite of where it started after 8 minutes? Express your answer as a common fraction.

Solution. $\frac{40}{2187}$ ■

Proposed by Joshua Pate.

Challenge 8. Square $ABCD$ has side length 100. An ellipse is inscribed in the square, touching sides AB, BC, CD, DA at E, F, G, H respectively. If $AE = AH = CF = CG = 36$, what is the area of the ellipse? Express your answer in terms of π.

Solution. Stretch the plane along the ellipse's shorter axis, and shrink so that the ellipse becomes a circle of radius 1. When this is done, the ratio of the area of the ellipse (which is now a circle) to the area of square $ABCD$ (which is still a rhombus but no longer a square) is unchanged. Now the circle is internally tangent to all four sides of the rhombus, and the ratios of the lengths of segments along a line are unchanged. Thus right triangle ABO, where O is the center of the circle, has altitude $AE = 1$ and hypotenuse divided in the ratio $\frac{AE}{BE} = \frac{36}{64} = \frac{9}{16}$. Since we also have $AE \cdot BE = 1$ because $\triangle AEO \sim \triangle OEB$, we have $AE = \frac{3}{4}$ and $BE = \frac{4}{3}$. Thus by the Pythagorean Theorem, $AO = \frac{5}{4}$ and $BO = \frac{5}{3}$. So the rhombus's diagonals have length $\frac{5}{2}$ and $\frac{10}{3}$ respectively, and its area is thus $\frac{25}{6}$. Meanwhile the circle's area is π, so the desired ratio is $\frac{6\pi}{25}$. Multiplying this by 100^2 (since that is the area of square $ABCD$ in the problem) gives the ellipse's area to be $\boxed{2400\pi}$. ■

Proposed by Matthew Kroesche.
Remark: Put on challenge round since it involves an ellipse

Challenge 9. Compute the number of integer pairs (x, y) such that $0 \le |x|, |y| < 10^{10}$ and

$$x^2 - 24y^2 = 1.$$

Solution. The main point is that $x^2-24y^2=1$ is a Pell equation. For now, we restrict our view to $x,y>0$, and we will add in the other solutions at the end. We can check that the least solution (under any reasonable ordering) will be $(5,1)$ because $y\geq 1$, so it follows that the solutions to $x^2-24y^2=1$ all take the form

$$x_n+y_n\sqrt{24}=\left(5+1\sqrt{24}\right)^n,$$

where n is some positive integer. This follows from theory on Pell equations, but it is not too hard to see that these are in fact solutions, for

$$x_n^2-24y_n^2=\left(x_n+y_n\sqrt{24}\right)\left(x_n-y_n\sqrt{24}\right)=\left(5+1\sqrt{24}\right)^n\left(5-1\sqrt{24}\right)^n=1.$$

Anyways, we note that $x=\sqrt{1+24y^2}>y$, so the only bounding condition we need to worry about is $x<10^{10}$. So we note that

$$x_n+y_n\sqrt{24}=\left(5+1\sqrt{24}\right)^n \qquad x_n-y_n\sqrt{24}=\left(5-1\sqrt{24}\right)^n,$$

from which we can solve

$$x_n=\frac{(5+\sqrt{24})^n+(5-\sqrt{24})^n}{2}.$$

We remark that, for the n we are worried about going over 10^{10}, the $(5-\sqrt{24})^n$ term will effectively vanish, so we have that $x_n\approx\frac{1}{2}(5+\sqrt{24})^n$.

Thus, we are roughly computing the number of positive integers n such that $x_n\leq 10^{10}$. Using our estimate for x_n, this is roughly equivalent to

$$n\leq\frac{\log\left(2\cdot 10^{10}\right)}{\log\left(5+\sqrt{24}\right)}.$$

Now we note that $5+\sqrt{24}\approx 5+5\approx 10$, so the bound on the right hand is roughly equivalent to

$$n\leq 10+\frac{\log 2}{\log 10}.$$

The $\frac{\log 2}{\log 10}$ is less than 1 and will not affect how many n are solutions. Namely, we will have a distinct solution for each $n\in\{1,\ldots,10\}$, giving 10 solutions with $x,y>0$.

We now remove the restriction that $x,y>0$. Each solution (x,y) will induce a solution where $x,y\geq 0$ by adding signs around. There are no solutions with $x=0$, for $-24y^2\leq 0<1$, and the only solutions with $y=0$ are $(\pm 1,0)$. So these add 2 solutions, and the remaining solutions with $x,y\neq 0$ are induced by signing the 10 solutions from earlier. This totals to $4\cdot 10+2=\boxed{42}$ total solutions. ■

Proposed by Nir Elber.

Remark: The lack of number theory in this section upset me, so I have risen from the grave to correct this tragedy.

Challenge 10. A certain tetrahedron has the property that all four of its faces are congruent. The tetrahedron has volume 2022, surface area 1200, and the sum of the lengths of its six edges is 160. If the centers of the inscribed circles of its four faces are joined to form a smaller tetrahedron, what is its volume? Express your answer as a common fraction.

Solution. We know the perimeter of each face is $\frac{160}{2}=80$, and the area of each face is $\frac{1200}{4}=300$. Thus, if we let each face have side lengths x,y,z, we know

$$x+y+z=80$$

and by Heron's formula

$$\sqrt{(x+y+z)(-x+y+z)(x-y+z)(x+y-z)}=1200$$

$$(-x+y+z)(x-y+z)(x+y-z)=18000$$

We also want to find a formula for the volume of a tetrahedron with all its faces congruent. (Note that the smaller tetrahedron also has all its faces congruent, so we will use this formula twice.) To do this, call the tetrahedron $ABCD$. We pick an edge AB, say with length x, and consider the two triangular faces ABC and ABD meeting at that edge. Their altitudes to AB both have length $h = \frac{2K}{x}$ (where K is the area of a face). Say the foot of an altitude divides the edge AB into two segments of length u and v. Then $u + v = x$, $u^2 + h^2 = y^2$, and $v^2 + h^2 = z^2$ (where y, z are the other two sides of the face). Thus $y^2 - z^2 = u^2 - v^2$ and the distance d between the feet of the altitudes from C and D is $d = |u - v| = \frac{|y^2 - z^2|}{x}$.

Now, we project into the plane containing one of these two altitudes (say, the altitude from C) and perpendicular to segment AB. The result is a triangle HCD', where H is the foot of the altitude from C to AB, and D' is the projection from D onto the plane. Thus $DD' = d$, $HC = HD' = h$, and $CD = AB = x$. Thus $CD' = \sqrt{x^2 - d^2}$. We want to find the length ℓ of the altitude from C in this triangle, since the foot of this altitude lies in the plane of face ABD, and so the volume V of the tetrahedron will then be $V = \frac{1}{3}K\ell$. Now $\ell h = m\sqrt{x^2 - d^2}$, where m is the altitude from H. Since the triangle is isosceles, the altitude from H is also a median, so we have $m^2 + \left(\frac{\sqrt{x^2 - d^2}}{2}\right)^2 = h^2$. Thus

$$\begin{aligned}
\ell &= \frac{m\sqrt{x^2 - d^2}}{h} \\
&= \frac{\sqrt{\left(h^2 - \frac{x^2 - d^2}{4}\right)(x^2 - d^2)}}{h} \\
&= \frac{\sqrt{\left(\frac{4K^2}{x^2} - \frac{x^4 - (y^2 - z^2)^2}{4x^2}\right)\left(x^2 - \frac{(y^2 - z^2)^2}{x^2}\right)}}{\frac{2K}{x}} \\
&= \frac{\sqrt{\left(16K^2 - x^4 + (y^2 - z^2)^2\right)\left(x^4 - (y^2 - z^2)^2\right)}}{4Kx} \\
&= \frac{\sqrt{\left((-x^4 - y^4 - z^4 + 2x^2y^2 + 2x^2z^2 + 2y^2z^2) - x^4 + (y^2 - z^2)^2\right)(x^2 + y^2 - z^2)(x^2 - y^2 + z^2)}}{4Kx} \\
&= \frac{\sqrt{(-2x^4 + 2x^2y^2 + 2x^2z^2)(x^2 + y^2 - z^2)(x^2 - y^2 + z^2)}}{4Kx} \\
&= \frac{\sqrt{(-x^2 + y^2 + z^2)(x^2 - y^2 + z^2)(x^2 + y^2 - z^2)}}{4K}
\end{aligned}$$

and so

$$V = \frac{\sqrt{2}}{12}\sqrt{(-x^2 + y^2 + z^2)(x^2 - y^2 + z^2)(x^2 + y^2 - z^2)} = 2022$$

Now to find the volume of the smaller tetrahedron, it suffices to find its side lengths x', y', z' in terms of x, y, z, since we can plug them into this formula. To do this, we consider again triangle HCD', with $HC = HD' = h$. Letting r be the radius of the inscribed circle of a face, so that $r = \frac{2K}{x+y+z} = \frac{15}{2}$, we consider the points R and S on sides HC, HD' respectively so that $HR = HS = r$. These two points are the projections of the centers of the inscribed circles of ABC, ABD respectively onto the plane of HCD'. The distance between them is easy to find, since $HRS \sim HCD'$, so we have

$$RS = CD' \cdot \frac{r}{h} = \frac{x \cdot CD'}{x + y + z} = \frac{\sqrt{x^4 - (y^2 - z^2)^2}}{x + y + z}$$

Meanwhile, the distance between the centers of the inscribed circles parallel to edge AB is also not hard to find. Since the distances from A, B, C to the points of tangency are $\frac{-x+y+z}{2}$, $\frac{x-y+z}{2}$, $\frac{x+y-z}{2}$ respectively (which follows from the fact that pairs of tangents from a point to a circle have the same length) we have that the distance between the points where the inscribed circles of ABC and ABD are tangent to AB is

$$\left|\frac{x - y + z}{2} - \frac{x + y - z}{2}\right| = |y - z|$$

So

$$\begin{aligned} x' &= \sqrt{RS^2 + |y-z|^2} \\ &= \sqrt{\frac{x^4 - (y^2 - z^2)^2}{x+y+z} + (y-z)^2} \\ &= \frac{\sqrt{x^4 - (y^2-z^2)^2 + (y-z)^2(x+y+z)^2}}{x+y+z} \\ &= \frac{\sqrt{x^4 + x^2y^2 - 2x^2yz + x^2z^2 + 2xy^3 - 2xy^2z - 2xyz^2 + 2xz^3}}{x+y+z} \end{aligned}$$

The values of y' and z', of course, are cyclic permutations of these. Next, to find the volume of the smaller tetrahedron, we find

$$\begin{aligned} -(x')^2 + (y')^2 + (z')^2 &= \frac{-x^4 + 2x^3y + 2x^3z - 2x^2yz - 2xy^3 - 2xy^2z - 2xyz^2 - 2xz^3 + y^4 + 2y^3z + 2y^2z^2 + 2yz^3 + z^4}{(x+y+z)^2} \\ &= \frac{-x^2(-x+y+z)^2 + (y^2+z^2)(-x+y+z)^2}{(x+y+z)^2} \\ &= \frac{(-x^2+y^2+z^2)(-x+y+z)^2}{(x+y+z)^2} \end{aligned}$$

where the factorization comes from completing the square in the numerator. Thus the volume of the smaller tetrahedron, by our formula, is

$$\frac{\sqrt{2}}{12}\sqrt{\frac{(-x^2+y^2+z^2)(-x+y+z)^2}{(x+y+z)^2} \cdot \frac{(x^2-y^2+z^2)(x-y+z)^2}{(x+y+z)^2} \cdot \frac{(x^2+y^2-z^2)(x+y-z)^2}{(x+y+z)^2}}$$

Plugging in our original expression for V, we see that this equals

$$\frac{V(-x+y+z)(x-y+z)(x+y-z)}{(x+y+z)^3} = \frac{V(s-x)(s-y)(s-z)}{s^3} = \frac{VK^2}{s^4}$$

where $s = \frac{x+y+z}{2}$ is the semiperimeter of a face. Since $s = 40$, $K = 300$, and $V = 2022$, the answer is

$$\frac{2022 \cdot 300^2}{40^4} = \frac{2022 \cdot 3^2}{4^4} = \boxed{\frac{9099}{128}}$$

■

Proposed by Matthew Kroesche.
Remark: Why, in the name of all that is holy, did I write this.

Tiebreaker Problems

Tiebreaker 1. What is the value of

$$\cfrac{4}{\sqrt{17+\cfrac{4}{\sqrt{17+\cfrac{4}{\sqrt{17+\cfrac{4}{\sqrt{\cdots}}}}}}}}?$$

Express your answer in simplest radical form.

Solution. Let x equal the answer. Then we have

$$x = \frac{4}{\sqrt{17+x}}$$

and thus

$$x\sqrt{x+17} = 4$$
$$x^2(x+17) = 16$$
$$x^3 + 17x^2 - 16 = 0$$

This cubic is easily seen to have -1 as a root, but the value of this number is clearly positive. Thus we divide out $x+1$ to get the quadratic

$$x^2 + 16x - 16 = 0$$

We solve this using the quadratic formula to get

$$x = \frac{-16 \pm \sqrt{16^2 + 4 \cdot 16}}{2} = \frac{\pm\sqrt{320} - 16}{2} = \pm\sqrt{80} - 8 = \pm 4\sqrt{5} - 8$$

Since $x > 0$, we answer $\boxed{4\sqrt{5} - 8}$. ■

Proposed by Matthew Kroesche.

Tiebreaker 2. At most how many of these five people can be telling the truth? (x is an integer.)

Aaron: $x < 3$
Baron: $x > -3$
Charon: $x^2 > 9$
Darren: $x^4 > 81$ and x is odd.
Erin: $x^4 > 81$ and x is even.

Solution. If $|x| < 3$, then Aaron and Baron are telling the truth but none of the others are. If $|x| > 3$, then Charon and either Darren or Erin are telling the truth, and exactly one of Aaron and Baron are as well. So, the answer is $\boxed{3}$. ■

Proposed by Maeve Dever.

Tiebreaker 3. What is $\frac{10^5-1}{41}$?

Solution. Of course, one may do this by direct computation to get the answer. Alternatively, one may get a good approximation by noting $41 \approx 40$, so the quotient should be about $\frac{10^5}{40} = \frac{10000}{4} = 2500$. The error is

$$2500 \cdot 41 - \left(10^5 - 1\right) = 2500 + 1 = 2501.$$

Now the direct computation $\frac{2501}{41} = 61$ is not too hard to do, or we could also note

$$\frac{2501 - 41}{41} = \frac{2540}{41} = 60,$$

which again gives $\frac{2501}{41} = 60 + 1 = 61$. So in total, we see that

$$\frac{10^5 - 1}{41} = 2500 - \frac{2501}{41} = 2500 - 61 = \boxed{2439}.$$

This is what we wanted. ■

Proposed by Joshua Pate.