2023 AMC Practice MATHCOUNTS Solutions Manual

Austin Math Circle

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Contents

Sprint Problems

Sprint 1. What is the value of $5^2 + 2^{(3^2)} - 7$?

Solution. Calculating each term in the sequence, we get $25 + 512 - 7 = \boxed{530}$ ■

Proposed by Swayam Gupta.

Sprint 2. Bob has a block of bouillon 20 feet wide, 168 inches long, and 5 yards high. What is its volume, in cubic feet?

Solution. The block is $20 * 14 * 15 = \boxed{4200}$ cubic feet. ■

Proposed by Pierce Lai.

Sprint 3. Princess Monika is leader of a pack of wolves. Suppose that seven wolves can crunch ten cabbages every hour. How many wolves would it take to crunch three cabbages every minute?

Solution. Three cabbages every minute is 180 cabbages every hour, so the answer is $18 * 7 = \boxed{126}$. ■

Proposed by Pierce Lai.

Sprint 4. What is the smallest square which is divisible by 120?

Solution. 120 has prime factorization $2^3 3^1 5$, so the answer is $120 * 30 = \boxed{3600}$. ■

Proposed by Pierce Lai.

Sprint 5. What is the median of the first 4 positive cubes? Express your answer as a decimal to the nearest tenth.

Solution. The median is the average of the middle two cubes, so $\frac{8+27}{2} = \boxed{17.5}$. ■

Proposed by Pierce Lai.

Sprint 6. Suppose a cube with side length s centimeters exists such that the length of its diagonal in centimeters and its surface area in square centimeters are numerically equal. What is its volume, in cubic centimeters? Express your answer as a common fraction in simplest radical form.

Solution. We must have $s\sqrt{3} = 6s^2$, and so $s = \frac{\sqrt{3}}{6}$. Hence, its volume is $\boxed{\frac{\sqrt{3}}{72}}$. ■

Proposed by Pierce Lai.

Sprint 7. Jimothy rolls a fair twenty-sided die twice. What is the probability that the first roll is greater than the second roll? Express your answer as a common fraction.

Solution. By fairness, the chance that the first roll is greater than the second is equal to the chance that it is less. Moreover, the sum of the three chances (first roll is less than, equal to, or greater than the second roll) must be 1. The chance that the first roll equals the second is $\frac{1}{20}$, so the answer is $\frac{1}{2}(1-\frac{1}{20}) = \boxed{\frac{19}{40}}$. ■

Proposed by Pierce Lai.

Sprint 8. In how many ways can Korra arrange four identical blue cups and three identical red cups in a line such that no two red cups are adjacent?

Solution. Starting with three red cups, Korra must put two blue cups between them. Then, there are two blue cups left over, and four distinguishable spaces to put them in (the space before the red cups, the spaces between the red cups, and the space after the red cups). Using stars and bars, this gives us $\binom{5}{3} = \boxed{10}$ ways. ■

Proposed by Pierce Lai.

Sprint 9. On Monday, Sally solves some math problems. On Tuesday, Sally solves three more problems than she did on Monday, and on Wednesday, Sally solves five more problems than she did on Tuesday. If she solved n problems in total, given that n is less than 30, what is the remainder when the sum of the possible values of n is divided by 6?

Solution. If Sally initially does x problems, the total amount of problems done is $x+(x+3)+(x+8) = 3x+11$. x can range from 1 to 6 (any higher, and 30 problems are exceeded). Note that the sum of the possible values of x is effectively $3(1+2+3+4+5+6)+11*6$. Since $1+2+3+4+5+6$ is odd, this is $\boxed{3}$ more than a multiple of 6. ■

Proposed by Josiah Kiok.

Sprint 10. Aang and his friends Baang and Caang are slurping ramen noodles. Suppose the ramen that Aang slurps costs \$8.55, the ramen that Baang slurps costs \$11.30, and the ramen that Caang slurps costs \$9.95. What proportion of the total cost is the ramen that Caang slurps? Express your answer as a common fraction.

Solution. The total cost is $8.55+9.95+11.30 = 29.80$. Notice that $9.95*3 = 29.85$, and so the gcd of 9.95 and 29.80 is 0.05. Hence, the answer is $\frac{9.95*20}{29.80*20} = \boxed{\frac{199}{596}}$. ■

Proposed by Pierce Lai.

Sprint 11. Suppose that $\triangle AMC$ is a triangle where $\angle A = 45°$ and $\angle C = 30°$. If the length of AM is $4\sqrt{2}$, what is the area of AMC? Express your answer in simplest radical form.

Solution. Let the altitude from M to AC be B. Then, ABM is a isoceles right triangle and CBM is a 30-60-90 triangle. This gives $MB = 4$, $AB = 4$, and $CB = 4\sqrt{3}$, so the answer is $\boxed{8+8\sqrt{3}}$. ■

Proposed by Pierce Lai.

Sprint 12. Suppose a, b are numbers such that their product equals 13 and their sum equals 10. What is the absolute value of their difference? Express your answer in simplest radical form.

Solution. We have that $ab = 13$ and $a + b = 10$, so $(a-b)^2 = a^2 + 2ab + b^2 - 4ab = 100 - 52 = 48$, and so the answer is $\boxed{4\sqrt{3}}$. ■

Proposed by Pierce Lai.

Sprint 13. On the planet of Eyjakoll, chickens have four legs and two wings, and cows have nine legs and five wings. Farmer John has some number of chickens and cows, such that the total number of legs is 252 and the number of wings is 134, how many chickens are there?

Solution. Let the number of chickens be x, and the number of cows be y. The question tells us that $4x + 9y = 252$ and $2x + 5y = 134$, or $y = 16$ and $x = \boxed{27}$. ■

Proposed by Pierce Lai.

Sprint 14. Square $ABCD$ has side length 10. Points X and Y lie on sides AB and BC respectively, such that $AX = XY = 6$. What is the area of $\triangle DXY$? Express your answer in simplest radical form.

Solution. We have $BX = 4$, and so $BY = \sqrt{20} = 2\sqrt{5}$ by the Pythagorean Theorem. Then $CY = 10 - 2\sqrt{5}$. So we have the areas

$$[DAX] = \frac{10 \cdot 6}{2} = 30$$

$$[BXY] = \frac{4 \cdot (2\sqrt{5})}{2} = 4\sqrt{5}$$

$$[CDY] = \frac{10 \cdot (10 - 2\sqrt{5})}{2} = 50 - 10\sqrt{5}$$

Thus the answer is

$$[ABCD] - [DAX] - [BXY] - [CDY] = 100 - 30 - 4\sqrt{5} - (50 - 10\sqrt{5}) = \boxed{20 + 6\sqrt{5}}$$

■

Proposed by Matthew Kroesche.

Sprint 15. Six people are playing a social deduction game. There are three rounds, and at the beginning of each round, Matthew shuffles a set of six distinct cards and deals one to each player. Given that Surya received the same card in all three rounds, what is the probability that Tarun also received the same card in all three rounds? Express your answer as a common fraction.

Solution. There is a $\frac{1}{5}$ chance that Tarun receives the same card as the first in each round following the first, since he cannot receive the card Surya had during the first round, but any of the other five are equally likely. Thus we answer $\left(\frac{1}{5}\right)^2 = \boxed{\frac{1}{25}}$. ■

Proposed by Matthew Kroesche.

Sprint 16. Let $a \bigoplus b = \frac{4(a+b)^2+2ab}{(2a+b)(a+2b)}$. What is the value of $(1 \bigoplus 2) \bigoplus (3 \bigoplus 4)$?

Solution. Notice that $a \bigoplus b = \frac{4a^2+8ab+4b^2+2ab}{2a^2+5ab+2b^2} = 2$ (at least, for $a \neq -2b$ and $a \neq -\frac{b}{2}$. Since all numbers are positive here, we should be fine). Hence, the answer is $\boxed{2}$. ■

Proposed by Pierce Lai.
Remark: This is hilarious. -Matthew

Sprint 17. Rick begins climbing up a stairwell at 3 steps per second. Meanwhile, Ashley climbs at 2 steps per second, but she changes her speed to 5 steps per second after she has climbed 20 steps. If both Rick and Ashley start climbing at the same time, and both reach the top of the stairwell at the same time, how many steps are in the stairwell?

Solution. Suppose there are n steps in the stairwell. Then Rick takes $\frac{n}{3}$ seconds to reach the top, and Ashley takes $\frac{20}{2} + \frac{n-20}{5} = \frac{n}{5} + 6$ seconds to reach the top. Setting these equal gives

$$\frac{n}{3} = \frac{n}{5} + 6$$

and we solve to get $n = \boxed{45}$ steps. ■

Proposed by Josiah Kiok.

Sprint 18. Alex, Isa, and Josiah are thirsty. They see that there happen to be five cups of fruit punch in front of them, and so each of them grabs one. However, unbeknownst to them, two of the fruit punch cups are actually filled with diluted Soylent instead of fruit punch. What is the expected total number of cups of real fruit punch that they grabbed? Express your answer as a common fraction.

Solution. Each person has a $\frac{3}{5}$ chance of grabbing a glass of real fruit punch, so by linearity of expectation the answer is $3 \cdot \frac{3}{5} = \boxed{\frac{9}{5}}$. ■

Proposed by Pierce Lai.

Sprint 19. Sanastasia trips while carrying a large barrel, containing a mixture of 50% toxic waste and 50% water, near a pond. The barrel's contents are dumped into the pond, causing a spill issue. After the spill, 0.01% of the liquid in the pond is toxic waste. What is the ratio of the original volume of the pond (before the spill) to the volume of the barrel?

Solution. Let the volume of the barrel be $2b$, and let the original volume of the lake be w. Because the toxic waste took up $\frac{1}{10000}$ of the lake's volume, we can equate the proportions as follows: $\frac{b}{w+2b} = \frac{1}{10000}$. Solving, we get $10000b = w + 2b$ and thus $w = 9998b$. Note that the volume of the barrel was selected to be $2b$ and not b, so the answer is $\frac{9998}{2} = \boxed{4999}$. ■

Proposed by Josiah Kiok.

Sprint 20. Equilateral $\triangle ABC$ has side lengths 6. Line l is parallel to BC and splits the triangle into two parts with equal areas. Given that l intersects AB and AC at X and Y, respectively, what is $\frac{AX}{XB}$? Express your answer in simplest radical form.

Solution. $\triangle ABC$ and $\triangle AXY$ are similar though AA similarity. Since the ratio of the area of $\triangle ABC$ to the area of $\triangle AXY$ is 2, the ratio of their sides is $\sqrt{2}$. Since $AB = 6$, $AX = 3\sqrt{2}$. Hence, $XB = 6 - 3\sqrt{2}$ and $\frac{AX}{XB} = \frac{3\sqrt{2}}{6-3\sqrt{2}}$. Rationalizing, we get $\frac{AX}{XB} = \boxed{\sqrt{2}+1}$ ■

Proposed by Swayam Gupta.

Sprint 21. How many positive integers less than 2023 have at least one 2 as a digit?

Solution. First, consider all the integers as four digit integers with leading 0s for convenience (hence $10 = 0010$). Second, since 0000 doesn't have 2 as a digit, we can consider that as well without changing the answer.

We will use complementary counting. Between 0000 and 0999, there are 9^3 integers that do not have 2 as a digit, and so there are $1000 - 729 = 271$ integers that do have 2 as a digit. Likewise, by symmetry there are 271 integers with 2 as a digit between 1000 and 1999. Finally, between 2000 and 2022 there are 23 integers and they all have 2 as a digit (the thousands place). Thus the answer is $271 * 2 + 23 = \boxed{565}$. ■

Proposed by Pierce Lai.

Sprint 22. The Colorizer takes a 2 by 5 grid and colors each cell either red, blue, or green. How many ways can the cells be colored such that no two adjacent cells share the same color?

Solution. Consider the recursive approach:

The base case (a single 2 by 1 column) has $3 * 2 = 6$ possibilities. There are $2x2 - 1 = 3$ ways to slap a column to the right of any 2 by x grid that satisfies the condition to create an 2 by $x+1$ grid. This is because there are two ways to pick the top of the column to avoid colliding with the left tile, and two ways to pick the bottom of the column – getting rid of the one case where both top and bottom are the same.

Hence, the answer is $6 * 3^4 = \boxed{486}$ ■

Proposed by Josiah Kiok.

Sprint 23. Let $\triangle ABC$ have sides AB, BC, and AC, with lengths 3, 4, and 5 respectively. Let D be the midpoint of BC, E be the midpoint of AC, and M be the midpoint of AD. What is the ratio of the area of $\triangle ABC$ to the area of $\triangle MED$?

Solution. We have

$$\frac{[\triangle ABC]}{[\triangle ACD]} \cdot \frac{[\triangle ACD]}{[\triangle AED]} \cdot \frac{[\triangle AED]}{[\triangle MED]} = 2 \cdot 2 \cdot 2 = \boxed{8}.$$

■

Proposed by Swayam Gupta.

Sprint 24. What is the greatest integer n such that 180! is divisible by 720^n?

Solution. Note that $720 = 2^4 3^2 5$. 180! has $90+45+22+11+5+2+1 = 176$ factors of 2, $60+20+6+2 = 88$ factors of 3, and $36+7+1 = 44$ factors of 5, and so the answer is $\boxed{44}$. ■

Proposed by Pierce Lai.

Sprint 25. Princess Syalis leaves her castle and begins strolling eastward at a leisurely pace of one mile per hour. At the same time, D'Whatsit leaves his house exactly sixty miles southeast of Syalis's castle, and begins jogging north at seven miles per hour. What is the closest that Syalis and D'Whatsit ever get to each other?

Solution. Put Syalis's castle at the origin. Thus at time t (where t is in hours), Syalis is at the point $(t,0)$, and D'Whatsit is at the point $(30\sqrt{2}, 7t-30\sqrt{2})$. Thus the distance between them is

$$D(t) = \sqrt{(t-30\sqrt{2})^2 + (7t-30\sqrt{2})^2} = \sqrt{50t^2 - 480t\sqrt{2} + 3600}$$

Completing the square gives

$$D(t) = \sqrt{50\left(t - \frac{24\sqrt{2}}{5}\right)^2 + 1296}$$

This takes on its minimum value at $t = \frac{24\sqrt{2}}{5}$ hours, and that minimum value is $\sqrt{1296} = \boxed{36}$ miles. ■

Proposed by Matthew Kroesche.

Sprint 26. What is the rightmost digit when $5^{38} - 3^{75}$ is written out in decimal form?

Solution. Since the units digit of 5^{38} is 5 and the units digit of 3^{75} is 7, the answer is just 8!

Right?

Normally, yes. However, the writer of this problem is evil. (Well, we changed the exponents to make it slightly less evil, but it's still evil.) We see that $3^{75} = 27 \cdot 9^{36} > 25 \cdot 5^{36} = 5^{38}$, so $5^{38} - 3^{75}$ is negative, and thus the units digit is $\boxed{2}$. ■

Proposed by Swayam Gupta.

Sprint 27. There are 100 fish in a pond. At the start of day 1, one fish is infected with a disease that affects fish gills. At the start of subsequent day, each living infected fish infects a healthy fish (if there is at least one healthy fish left to be infected). Then, all fish which were infected two days ago die from gill issues. Assuming the number of fish does not increase or decrease through means other than the disease, on which day will there be no more living fish in the pond for the first time?

Solution. Let $f(x)$ be the number of infected fish on day x. Note that the number of infected fish on a given day is simply twice the number of fish infected yesterday minus the fish infected the day before, which can be represented as $f(x) = 2f(x-1) - f(x-2)$. One easily finds that $f(1) = 1$, $f(2) = 2$, $f(3) = 3...$ and can easily conjecture (and prove, via induction) that $f(x) = x$. Let $g(x)$ be the number of dead fish on day x. This is equal to the total number of the fish infected at least two days prior – and since $f(x) = x$, this is simply $g(n) = \frac{(n-2)(n-1)}{2}$ by Gaussian summations. The day that this value exceeds 100 is $n = 16$, and hence the answer is $\boxed{16}$.

■

Proposed by Josiah Kiok.
Remark: Skill issue.

Sprint 28. Nathan randomly picks a positive integer divisor d of 2024. Then, Michael and Morghan each randomly and independently pick a positive integer divisor of d. What is the probability that Michael and Morghan both pick the same number? Express your answer as a common fraction.

Solution. The number $2024 = 2^3 \cdot 11 \cdot 23$ has $4 \cdot 2 \cdot 2 = 16$ positive integer divisors. Thus the answer is $\frac{1}{16}$ times the sum of the reciprocal of the number of divisors of each one of these divisors. Since the divisor counting function is multiplicative, we can write this sum as

$$\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{2}\right) = \frac{25}{12} \cdot \frac{3}{2} \cdot \frac{3}{2} = \frac{75}{16}$$

and thus we answer $\boxed{\frac{75}{256}}$. ■

Proposed by Matthew Kroesche.

Remark: Solving this by listing out all 16 divisors of 2024 is possible, but much more time consuming. This problem is meant to illustrate the usefulness of thinking multiplicatively.

Sprint 29. Let $ABCDEF$ be a concave hexagon where all angles are either 90° or 270° and all sides have positive integer side length. Let the perimeter of $ABCDEF$ be 12. What is the ratio of the largest possible area of $ABCDEF$ to the smallest possible area of $ABCDEF$? Express your answer as a common fraction.

Solution. For the purpose of clarity, rotate and reflect until $ABCDEF$ is a giant L (where A is the top left point, B is the top right, etc.). Note that the perimeter of $ABCDEF$ is equivalent to $2(AF + ED)$, which means that once AF and ED are chosen, the other side lengths can be freely adjusted for the most part. Thus, $AF + ED = 6$. The area is equivalent to $AF * FE - BC * CD$ by rectangle subtraction. Note that the area is minimized when $AB = DE = 1$ – otherwise, setting those values to 1 always decreases the area while leaving the perimeter the same. This leads to a shape where $AF = FE = 3$, and the area is 5 (note all other values of AF and FE are possible but still give an area of 5). Note that the area is maximized when $BC = CD = 1$ for the opposite reason. $BC * CD$ is then always 1. $AF * FE$ is maximized at $3 * 3 = 9$, so the maximal area is 8.

Hence, the answer is $\boxed{\frac{8}{5}}$. ■

Proposed by Josiah Kiok.

Sprint 30. Six people are playing a social deduction game. There are three rounds, and at the beginning of each round, Matthew shuffles a set of six distinct cards and deals one to each player. Given that Ryan received a different card in all three rounds, what is the probability that Kevin also received a different card in all three rounds? Express your answer as a common fraction.

Solution. Count the number of ways to give Ryan and Kevin each a card, three times, such that Ryan never gets the same card twice. This is $6 \times 5 \times 4 \times 5^3$, where the $6 \times 5 \times 4$ counts the number of ways to give Ryan a different card each time, and the 5^3 counts the number of ways to give Kevin any card other than the one Ryan gets that round. This is the denominator of our fraction. The numerator is the number of ways to give them each a card such that both Kevin *and* Ryan get a different card each time. For the first game, there are 6×5 ways to give Kevin and Ryan each a card. For the second, we split into cases depending on how many new cards are seen.

- *Case 1:* Kevin and Ryan each receive a card different from either of the two that were dealt in the first round. Then there are 4×3 ways to deal the cards in the second round. In the third round, the number of ways to deal the cards is $(2 \times 3) + (2 \times 4) = 14$, where the 2×3 is the case where Ryan is given one of the two cards that neither of them have had before, and the 2×4 is the case where Ryan is given one of the two cards that Kevin has had before. In total, this gives $4 \times 3 \times 14 = \mathbf{168}$ ways.
- *Case 2:* One of Kevin and Ryan receives the same card that the other received in the first round, and the other receives a new card. Then there are 2×4 ways to deal the cards in the second round, since we have to choose whether Kevin or Ryan gets a card that has been seen before, then we have to choose one of the four new cards to give to the other one. Then in the third round, there is one card that is completely off limits (because

both Kevin and Ryan have seen it) and two more that exactly one of them have seen. Thus the number of ways to deal is $(3 \times 3) + 4 = 13$, where the 3×3 is the case where Ryan is given one of the three cards that neither of them have had before, and the 4 is the case where Ryan is given the card that Kevin has had before but Ryan has not. In total, this gives $2 \times 4 \times 13 = \mathbf{104}$ ways.

- *Case 3:* Each of Kevin and Ryan receives the same card that the other received in the first round. Then there are two cards that have each been used twice, and neither may be used again. So there are just $4 \times 3 = \mathbf{12}$ ways to choose which cards each of them gets in the third round.

Thus the answer is

$$\frac{6 \times 5 \times (168 + 104 + 12)}{6 \times 5 \times 4 \times 5^3} = \boxed{\frac{71}{125}}$$

■

Proposed by Matthew Kroesche.

Remark: This answer comes out to 56.8%. Without the information that Ryan got a different card in every round, it's only $\frac{5}{9} = 55.\overline{5}\%$. It makes sense that if one person got a different card each time, another person is slightly more likely to as well. -Matthew

Target Problems

Target 1. Iceberg Arrowhead is $6/7$ as old as Aratak, while Iceberg's distant relative, Heisenberg Arrowhead, is 8 times as old as Aratak. If Heisenberg is 100 years older than Iceberg, what is the positive difference between the ages of Iceberg and Aratak?

Solution. Let i, a, h be the ages of Iceberg, Aratak, and Heisenberg respectively. We have

$$h = 8a$$

$$i = \frac{6}{7}a$$

$$h = i + 100$$

Substituting gives

$$8a = \frac{6}{7}a + 100 \implies \frac{50}{7}a = 100 \implies a = 14$$

Then Iceberg is 12 years old, so the age difference is $\boxed{2}$ years. ■

Proposed by Justin Xiao.

Target 2. Mathew, Matthew, and Mattthew are playing the hit game Dungeons, Dungeons, and More Dungeons. At each turn in the game, the player with the most t's in their name will be nominated sacrifice, and remove one of the t's from their name, duplicate it, and give one t to each of the two other players. If there are multiple players with the most number of t's, then the player who was least recently sacrificed will become the new sacrifice. After the end of one hundred turns of this game, how many t's are in the name of the player originally named Matthew?

Solution. The first few turns go Mathew, Matthew, Mattthew → Matthew, Mattthew, Matthew → Mattthew, Matthew, Mattthew → Matthew, Mattthew, Matttthew. Hence, we see that the pattern repeats every three turns, so the answer is $3 + 33 = \boxed{36}$ t's. ■

Proposed by Pierce Lai.

Target 3. Suppose Allefonz has a strictly-increasing geometric sequence, consisting of 4 terms that are integers between 100 and 400 inclusive. What is the least possible value of the last term in the sequence?

Solution. Let the ratio between terms in the series be r. Since the terms are integral, r must be a rational number, so let it be expressed in lowest terms as $\frac{p}{q}$. Now, since we just want the least possible value of the last term, and $r > 1$, we must have $p = q + 1$, since otherwise we could create a sequence with lower last term by starting with the same first term. In addition, the first term must be divisible by q^3. Testing $r = 3/2$ gives $104, 156, 234, 351$; testing $r = 4/3$ gives $108, 144, 192, 256$; testing $r = 5/4$ gives $128, 160, 200, 250$; testing $r = 6/5$ gives $125, 150, 180, 216$. Clearly, any higher value of q would be worse, so the answer is $\boxed{216}$. ■

Proposed by Pierce Lai.

Target 4. Bob has some (positive integer) number of whole fried cows. He counts his whole fried cows, and says, "If I had two more whole fried cows, or if I had one more person in my family, I could give each person in my family the same whole number of whole fried cows." Given that the number of people in Bob's family is at least 10, what is the least possible number of whole fried cows Bob has?

Solution. Let the number of people in Bob's family be n and the number of whole fried cows be c. We must have that $n|c+2$ and $n+1|c$. From the second equation, we know that $c = m(n+1)$ for some positive integer m. Then, n divides $mn+m+2$, so n divides $m+2$. The smallest value of m for which this is possible in terms of n is $m = n-2$. Hence, the smallest number of cows is given by $n = 10$ and $m = 8$, for $c = \boxed{88}$. ■

Proposed by Pierce Lai.

Target 5. Qui-Gon Jinn is playing a game of Mario Kart. However, his mortal enemy, Coconut Maul, has rigged his console to have a $\frac{1}{n^2+2n}$ chance of crashing whenever he finishes in n-th place. If Qui-Gon knows he will finish anywhere from 6th to 13th inclusive with equal probability, what is the probability that his console will not crash? Express your answer as a common fraction.

Solution. The probability that Qui-Gon Jinn's console crashes is

$$\frac{1}{8}\sum_{n=6}^{13}\frac{1}{n^2+2n} = \frac{1}{16}\sum_{n=6}^{13}\left(\frac{1}{n}-\frac{1}{n+2}\right) = \frac{1}{16}\left(\frac{1}{6}+\frac{1}{7}-\frac{1}{14}-\frac{1}{15}\right) = \frac{1}{16}\left(\frac{1}{10}+\frac{1}{14}\right) = \frac{1}{16}\cdot\frac{24}{140} = \frac{3}{280}$$

Thus we answer $\boxed{\dfrac{277}{280}}$. ■

Proposed by Justin Xiao.

Target 6. A flying spider and a spidery fly are playing ping pong in a flying spidery submarine. The score starts at 0-0, and after 12 matches end up tied at 6-6. Suppose that at no time is the product of the two players' scores prime. What is the number of possible sequences of scores?

Solution. If the product of the scores is prime, then the score is 1-2, 1-3, or 1-5 in some order. Note also that the score can never be 1-1 or else the next point would make it 1-2 or 2-1. So the first player to score has to score at least 4 times in a row. WLOG suppose the flying spider scores first (we multiply by 2 at the end to account for this). Then if the flying spider scores six consecutive points, the next six points must all be scored by the spidery fly to tie it up. There is **1** way to do this. Otherwise, the flying spider scores exactly four consecutive points, then the spidery fly scores at least twice. (If it only scores once, then the next point scored by the flying spider will make the score 5-1.) In this scenario, the score is 4-2. The flying spider needs to score two of the next six points and the spidery fly needs to score the other four. They can happen in any order, so there are $\binom{6}{2} = 15$ ways for this to happen. Thus there are $1+15 = 16$ possible sequences of scores that involve the flying spider getting the first point, and 16 where the spidery fly gets the first point, for a total of $16+16 = \boxed{32}$ sequences. ■

Proposed by Pierce Lai.

Target 7. There are six targets, which once hit, score 1, 2, 3, 4, 5, and 6 points, respectively. Violet throws a boomerang and hits some number of the targets once, scoring 10 points. How many distinct sets of targets could she have hit?

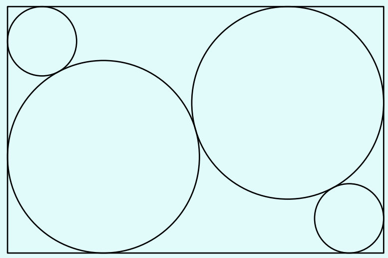
Solution. Assume 6 was hit. The only combinations which total 6 are $6+4$ and $6+3+1$.

Assume 6 was not hit. Then, of the remaining targets, Violet must have missed 5 points. The only combinations which total this are 5, $4+1$, and $3+2$.

Overall, there are only $\boxed{5}$ sets of targets that work. ■

Proposed by Josiah Kiok.

Target 8. Four circles with non-overlapping interiors are drawn inside a rectangle. Two of circles have radius 25, and the other two have radius 9. The two circles of radius 25 are each tangent to a pair of adjacent sides of the rectangle (and there is no side that is tangent to both the circles of radius 25), and also tangent to each other. The two circles of radius 9 are each tangent to a pair of adjacent sides of the rectangle, and also tangent to one of the circles of radius 25. What is the area of the rectangle?



Solution. First, we figure out the length of the vertical sides. If we draw the radii from both circles on the left side to the points of tangency with the left side, and if we also draw the radii to the point where those two circles are tangent to each other, this gives a trapezoid with two right angles, bases of length 9 and 25, and the side opposite the two right angles having length $9+25=34$. We can split this trapezoid into a rectangle and a right triangle with hypotenuse 34 and one leg of length $25-9=16$. Then the other leg, and thus the fourth side of the trapezoid, has length $\sqrt{34^2-16^2}=30$. So the vertical sides of the rectangle have length $30+25+9=64$.

Now to find the longer sides of the rectangle, we connect the centers of the two large circles. The line segment joining the centers of the circles has length $25+25=50$, and the vertical distance between the centers of the circles is $64-25-25=14$. Then the horizontal distance between the centers of the circles is $\sqrt{50^2-14^2}=48$. Thus the horizontal sides of the rectangle have length $25+25+48=98$, and so the area is $64\cdot 98=\boxed{6272}$. ■

Proposed by Matthew Kroesche.

Team Problems

Team 1. Boba doesn't like how certain numbers are named, and so he shuffles the names of the integers from 1 to 4. Under his renaming, he says "Four minus Three is One, Two plus Three equals One plus Four, and One Times Four equals Two plus One." What is the value of One times Two plus Three?

Solution. The first clue tells us that either Four equals Three plus One, and, substituting into the second clue gives us Two plus Three equals Three plus 2 * One, or Two equals 2* One. Hence, One is either 1 or 2. If One equaled 1, then we would get One, Two, Three, Four equals 1, 2, 3, 4 from the second clue, which fails the third clue. Hence, One equals 2, and so Two equals 4, Three 1, and Four 3. (We see that this satisfies the third clue.) Hence, the answer is $2*4+1 = \boxed{9}$. ■

Proposed by Pierce Lai.

Team 2. Jxiao flips a fair coin five times. What is the probability that some three consecutive flips out of those five are Heads, Tails, Heads in that order? Express your answer as a common fraction.

Solution. There are $2^5 = 32$ possible outcomes for five coin flips. We count how many of them contain HTH. There are four of the form HTH??, four of the form ?HTH?, and four of the form ??HTH. However, the outcome HTHTH was counted twice, so we subtract one. Thus the probability is $\frac{4+4+4-1}{32} = \boxed{\frac{11}{32}}$. ■

Proposed by Pierce Lai.

Team 3. A cone has height 10 and volume 60π. What is the side length of the largest cube that will fit inside it, if one of its faces lies in the same plane as the base of the cone? Express your answer as a common fraction.

Solution. The cone has $\frac{1}{3}\pi r^2 \cdot 10 = 60\pi$, so $r = 3\sqrt{2}$. Now place the cube so one of its faces is centered on the base of the cone, and each of its top four vertices are along the curved surface of the cone. Then if the cube has side length s, the radius of the cross-sectional area of the cone containing the top face of the cube is $3\sqrt{2}\left(1-\frac{s}{10}\right)$, and we set this equal to $\frac{s\sqrt{2}}{2}$. Clearing denominators,

$$30 - 3s = 5s$$

and thus $s = \boxed{\frac{15}{4}}$. ■

Proposed by Matthew Kroesche.

Team 4. Hughes is playing a game of Metaphorical Pail Soccer. Each minute, he kicks a bucket by exactly one meter in a direction randomly chosen between north, south, east and west. After 4 minutes, what is the probability that the bucket is exactly 2 meters away from where it started? Express your answer as a common fraction.

Solution. There are $4^4 = 256$ possible outcomes after 4 minutes have passed. If the bucket is exactly 2 meters away from where it started (which is WLOG the origin on the coordinate plane) then it is either at $(\pm 2, 0)$ or $(0, \pm 2)$. Suppose it is at $(2,0)$. Then the sequence of Hughe's four kicks must have been EEEW or EENS in some order. There are 4 permutations of the first sequence and 12 permutations of the second, for a total of 16 sequences that cause the bucket to end up at $(2,0)$. Similarly, there are 16 sequences for each of the other three points, for a total of 64. Thus the answer is $\frac{64}{256} = \boxed{\frac{1}{4}}$. ■

Proposed by Justin Xiao.

Team 5. In acute triangle $\triangle ABC$, D is the foot of the altitude from A to side BC, E is the foot of the altitude from D onto side AC, and F is the foot of the altitude from D onto side AB. If $AD = 25$, $DE = 15$, and $DF = 7$, what is the area of $\triangle ABC$? Express your answer as a common fraction.

Solution. Since $\triangle ADE \sim \triangle ABD$ and $\triangle ADF \sim \triangle ACD$, we have

$$BD = \frac{AD \cdot DE}{AE} = \frac{25 \cdot 15}{20} = \frac{75}{4}$$

$$CD = \frac{AD \cdot DF}{AF} = \frac{25 \cdot 7}{24} = \frac{175}{24}$$

$$BC = BD + CD = \frac{75}{4} + \frac{175}{24} = \frac{625}{24}$$

Then the area of $\triangle ABC$ is

$$\frac{AD \cdot BC}{2} = \boxed{\frac{15625}{48}}$$

■

Proposed by Matthew Kroesche.

Team 6. Balice and Ob are reading a book called "A Brief History of the Emergent Methods of Fleshly Becoming" for their synthetic biology class. Suppose Balice and Ob both begin reading at 12:00 AM, and each read continuously at their own steady rate without stopping. Suppose furthermore that Ob, having already read the first few sections, begins at the top of page 101, while Balice begins reading at the top of page 1. Suppose that at precisely 8:00 AM, Balice catches up to Ob in terms of reading, and that the total number of pages they have read since 6:00 AM is exactly 225. If Balice finishes reading the book at 8:00 PM, what time will it be when Ob finishes reading?

Solution. Suppose that, at the instant Balice catches up to Ob, they have each just finished reading page n. Then the total number of pages they have read so far is $n + (n - 100) = 2n - 100$, and they read a quarter of those pages in the two hours from 6:00 AM to 8:00 AM. Thus $\frac{2n-100}{4} = 225$, so $2n - 100 = 900$ and thus $n = 500$. Thus Balice reads $\frac{500}{4} = 125$ pages every two hours, and Ob reads $\frac{400}{4} = 100$ pages every two hours. If it takes Balice 12 additional hours to finish the book, then there must be 750 pages left. It will take Ob 15 hours to read those 750 pages, so Ob will finish at $\boxed{\text{11:00 PM}}$. ■

Proposed by Pierce Lai.

Team 7. A triangle with integer side lengths has perimeter 80 and area 180. What is the product of its side lengths?

Solution. By Heron's formula, we have

$$\sqrt{40(40-a)(40-b)(40-c)} = 180$$

Thus, letting $x = 40 - a$, $y = 40 - b$, $z = 40 - c$ be positive integers, we have

$$40xyz = 180^2$$

and so

$$xyz = 810$$

We also have

$$x+y+z=(40-a)+(40-b)+(40-c)=120-(a+b+c)=120-80=40$$

Now consider how many of x, y, z can be divisible by 3. At least one is divisible by 3, because the product is divisible by 3. But at least one is not, because the sum is not. Moreover, exactly one cannot be divisible by 3, or else that one would have to be divisible by $3^4 = 81$, which is too big. So exactly two are divisible by 3. Moreover, since the product of those two is divisible by 81, either they are both divisible by 9, or one of them is divisible by 27. If they are both divisible by 9, then we have $9u \cdot 9v \cdot z = 810$ and thus $uvz = 10$, but also $9u + 9v + z = 40$. The second condition requires that $z \equiv 4 \pmod 9$, but no divisor of 10 is 4 mod 9, so this is impossible. Thus one of x, y, z must be divisible by 27 after all, and thus it must exactly equal 27 because 54 is too big. WLOG $x = 27$. Then $y + z = 13$ and $yz = 30$, so we quickly see that y, z are $10, 3$ in some order. So then a, b, c are $40-27 = 13$, $40-10 = 30$, and $40-3 = 37$, and their product is $\boxed{14430}$. ■

Proposed by Matthew Kroesche.

Team 8. Lea and Shizuka are dueling. The duel consists of a series of rounds, each of which is equally likely to be won by either player, independently of all other rounds. (No round can end in a tie.) The duel ends as soon as a round ends and $2\ell + s \geq 12$, where ℓ is the total number of rounds won by Lea and s is the total number of rounds won by Shizuka. What is the probability that Shizuka wins strictly more rounds than Lea during their duel? Express your answer as a common fraction.

Solution. I'm going to come back and revisit this later, since I think there's some really cool math going on behind the scenes. In the meantime, this can be solved using complementary counting, since there's only four ways for Lea to win (5-2, 6-0, 5-3, 6-1) and the last two require her to win her final match (or else it would have ended earlier). There's also one way for Lea and Shizuka to tie, 4-4. So the probability that Lea wins is

$$\frac{21}{128} + \frac{1}{64} + \frac{35}{256} + \frac{6}{128} = \frac{93}{256}$$

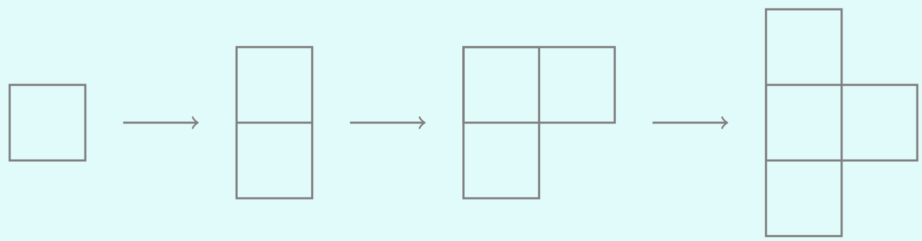
and the probability that they tie 4-4 is $\frac{70}{256}$. So the answer turns out to be

$$1 - \frac{93}{256} - \frac{70}{256} = \boxed{\frac{93}{256}}$$

This suggests that Lea's probability of winning is the same as Shizuka's, in spite of the asymmetry in the rules. Based on Python calculations, if you replace 12 with an integer n, this always remains true *unless* n is 2 mod 3, in which case Lea seems to always win with probability $\frac{1}{2}$. So there's probably some cool bijection going on, but I don't yet see what it is. ■

Proposed by Matthew Kroesche.

Team 9. Belsy Fur, an evil sentient blue square, has a diabolical plan to take over the world where he sews clones of himself together into a giant tapestry. At step 1, Belsy retrieves a single unit square clone of himself to start as the beginning of the tapestry. At step 2, he randomly picks an exposed edge of one of the unit squares of the tapestry, and then sews another unit square clone along that length. For step 3, he again randomly picks an exposed edge of a unit square and then sews another unit square, and so forth. An example sequence of generated tapestries might look like



At the end of step 5, what is the expected value of the perimeter of the tapestry? Express your answer as a common fraction.

Solution. At the end of step 3, we have a $\frac{2}{3}$ chance of having an L-shaped triomino, and a $\frac{1}{3}$ chance of having an I-shaped triomino, depending on which of the six edges we choose. After step 4, we have the following possibilities:

- An I piece happens with probability $\frac{1}{3} \times \frac{1}{4} = \frac{1}{12}$, since we have to pick one of two specific edges of an I-shaped triomino.
- An O piece happens with probability $\frac{2}{3} \times \frac{1}{4} = \frac{1}{6}$, since we have to pick one of two specific edges of an L-shaped triomino.
- An S/Z piece happens with probability $\frac{2}{3} \times \frac{1}{4} = \frac{1}{6}$, since we have to pick one of two specific edges of an L-shaped triomino.
- A T piece happens with probability $\frac{1}{3} \times \frac{1}{4} + \frac{2}{3} \times \frac{1}{4} = \frac{1}{4}$, since we can build it from either type of triomino.
- An L/J piece happens with probability $\frac{1}{3} \times \frac{1}{2} + \frac{2}{3} \times \frac{1}{4} = \frac{1}{3}$, since we can build it from either type of triomino.

Every time Belsy Fur sews a new piece on, the perimeter of the tapestry increases by 2, *unless* he is filling in a corner, in which case it stays the same. (If he keeps sewing beyond step five, it is possible that it could decrease too, by filling in a hole that already has three or more edges.) Thus, all of these tetrominoes have perimeter 10 except for the O piece, which only has 8 because a corner was filled in on the fourth step. Thus we consider each case:

- If step 4 ended with an I piece, the new pentomino will have perimeter 12 since there are no corners to fill in.
- If step 4 ended with an O piece, the new pentomino will have perimeter 10 since there are no corners to fill in.
- If step 4 ended with an S/Z/T piece, the new pentomino will have perimeter 12 with probability $\frac{3}{5}$ and 10 with probability $\frac{2}{5}$, since four of the ten exposed edges are inside corners. Thus the expected perimeter in this case is $\frac{56}{5}$.
- If step 4 ended with an L/J piece, the new pentomino will have perimeter 12 with probability $\frac{4}{5}$ and 10 with probability $\frac{1}{5}$, since two of the ten exposed edges are inside corners. Thus the expected perimeter in this case is $\frac{58}{5}$.

All in all, the expected perimeter should be

$$\frac{1}{12} \cdot 12 + \frac{1}{6} \cdot 10 + \frac{1}{6} \cdot \frac{56}{5} + \frac{1}{4} \cdot \frac{56}{5} + \frac{1}{3} \cdot \frac{58}{5} = \boxed{\frac{56}{5}}$$

■

Proposed by Pierce Lai.

Team 10. In the addition problem

$$\begin{array}{cccccc} & S & K & I & L & L \\ + & I & S & S & U & E \\ \hline & L & E & E & K & S \end{array}$$

each letter represents a different nonzero base ten digit. What is the value of the four-digit number LUKE?

Solution. First, exactly one of the 10^2 and 10^4 columns can have a one carried into it, because the L and the E stand for different digits. Second, regardless of which it is, the 10^3 column cannot have a one carried into it, or else $I+S \geq 10$ (because the E cannot stand for zero) and the result of the sum would be a six-digit number.

First, suppose there is a one carried into the 10^4 column (but not the 10^3 or 10^2). Then

$$\begin{aligned} 1+S+I &= L \\ K+S &= 10+E \\ I+S &= E \end{aligned}$$

This is no good because $K-I=10$ but K and I are both digits. So suppose instead that there is a one carried into the 10^2 column (but not the 10^3 or 10^4). Then

$$\begin{aligned} S+I &= L \\ K+S &= E \\ 1+I+S &= E \end{aligned}$$

Thus $E=L+1$ and $K=I+1$. Now we split into cases based on whether there is a one carried into the 10^1 column. If there is not, then

$$\begin{aligned} L+E &= S \\ L+U &= 10+K \end{aligned}$$

This is no good because $L=S+I=L+E+I$, so $E+I=0$. So suppose instead that there is a one carried into the 10^1 column. Then

$$\begin{aligned} L+E &= 10+S \\ 1+L+U &= 10+K \end{aligned}$$

Then $L=S+I=L+E+I-10$, so $I+E=10$. Since $E=L+1$, $L+I=9$. So we write

$$\begin{aligned} I &= 9-L \\ E &= L+1 \\ K &= 10-L \\ S &= 2L-9 \\ U &= 19-2L \end{aligned}$$

Additionally, these all stand for different nonzero digits. Since $2L-9$ is a digit, we have $L \geq 5$. Since $L+1$ is a digit, we have $L \leq 8$. We rule out $L=5$ because it would cause K to equal 5 as well; $L=6$ because because it would cause $I=S=3$; and $L=7$ because it would cause $S=U=5$. Thus we have $L=8, I=1, E=9, K=2, S=7, U=3$. (As a check, the addition problem would read $72188+17739=89927$, which is true.) So $LUKE=\boxed{8329}$. ∎

Proposed by Matthew Kroesche.

Countdown Problems

Editor's note: The countdown questions were randomized on the day of the contest, so apart from question 0, the order in which they appear here does not match the order in which they appeared on the contest.

Countdown 0. Grampton St. Rumpterfrabble and his friend Saunterblugget Hampterfuppinshire the irascible coxswain are watching the classic film series *The Duchess Approves* while eating copious amounts of Smile Dip. Suppose that the film series originally had 20 episodes and Grampton St. Rumpterfrabble and his friend Saunterblugget Hampterfuppinshire the irascible coxswain together eat a total of 400 packets of Smile Dip across the entire series. Suppose further that the amount of Smile Dip Grampton St. Rumpterfrabble and his friend Saunterblugget Hampterfuppinshire the irascible coxswain ate per episode forms an arithmetic series, and that Grampton St. Rumpterfrabble and his friend Saunterblugget Hampterfuppinshire the irascible coxswain ate 39 packets of Smile Dip during the 20th episode. How many packets of Smile Dip did Grampton St. Rumpterfrabble and his friend Saunterblugget Hampterfuppinshire the irascible coxswain eat during the third episode?

Solution. Suppose Grampton St. Rumpterfrabble and his friend Saunterblugget Hampterfuppinshire ate a packets of Smile Dip on the first day, and r more packets of Smile Dip on each successive day than the previous day. Then we have $a+19r=39$, and

$$a+(a+r)+(a+2r)+\cdots+(a+19r)=20a+190r=400$$

So $2a+19r=40$, and subtracting, $a=1$ and $r=2$. Then our answer is $a+2r=\boxed{5}$ packets. ■

Proposed by Pierce Lai.

Countdown 1. Bradley is selling Brad Buns™. Suppose that a group of seven people can consume seventy Brad Buns™ in seven hours and seven hundred seventy minutes. How long in minutes would it take a group of seventeen people to consume seventy seven Brad Buns™?

Solution. Seven hours and seven hundred and seventy minutes is 1190 minutes. Hence, it would take seventeen people $(1190*7/17*77/70)=\boxed{539}$ minutes to consume seventy seven Brad Buns™. ■

Proposed by Pierce Lai.

Countdown 2. How many two-digit numbers are divisible by 6 and have their digits in nondecreasing order?

Solution. This is simple enough to bash by hand. The numbers are $12, 18, 24, 36, 48, 66$, and 78, so the answer is $\boxed{7}$. ■

Proposed by Pierce Lai.

Countdown 3. Blerb Bobbert buys 3 boxes and 2 bins of bobbers. Each bin Blerb buys contains 4 bobbers and each box Blerb buys contains 3 bobbers plus two bins of bobbers. If Blerb began with 5 bobbers, how many bobbers does Blerb have now?

Solution. Each box contains $3+2*4=11$ bobbers, so Blerb now has $5+3*11+2*4=\boxed{46}$ bobbers. ■

Proposed by Pierce Lai.

Countdown 4. Jebediah has a large quantity of cones. Suppose he has three types of cones, long, medium, and short, in a ratio of 33:17:1. If Jebediah has a total of 2499 cones, how many medium cones does he have?

Solution. Notice that exactly $\frac{17}{33+17+1} = \frac{1}{3}$ of Bob's cones are medium. Thus the answer is $2499/3 = \boxed{833}$. ■

Proposed by Pierce Lai.
Remark: Reference to cone cells.

Countdown 5. Pierce is flipping Matthews. When he flips a Matthew, the Matthew can land on heads, feets, or body. Overall, a Matthew has a 20% chance of landing on heads and a 40% chance of landing on feets. If Pierce flips a Matthew 3 times, what is the probability that he gets the same landing all three times? Express your answer as a common fraction.

Solution. Notice that the chances are $\frac{1}{5}$, $\frac{2}{5}$, and $\frac{2}{5}$. Hence, the answer is $\frac{1^3+2^3+2^3}{5^3} = \boxed{\frac{17}{125}}$. ■

Proposed by Pierce Lai.

Countdown 6. Ethelred rolls two fair six-sided dice. What is the probability that the product of the numbers showing is at least 12? Express your answer as a common fraction.

Solution. The pairs which satisfy this are $(2,6),(3,4),(3,5),(3,6),(4,4),(4,5),(4,6),(5,5),(5,6),(6,6)$, so the answer is $\frac{14+3}{36} = \boxed{\frac{17}{36}}$. ■

Proposed by Pierce Lai.

Countdown 7. What is the volume of a cube (in cubic centimeters) that has surface area 150 square centimeters?

Solution. The cube has side length 5 cm, and thus has volume $\boxed{125}$ cubic centimeters. ■

Proposed by Pierce Lai.

Countdown 8. Rich is juggling three large balls. Each of those large balls are juggling 4 smaller balls. How many more balls will he need if he wants to instead juggle 4 large balls that each themselves are juggling 5 small balls?

Solution. There are $3+3*4 = 15$ balls to begin with, and Rich wants $4+4*5 = 24$ balls at the end, so the answer is $\boxed{9}$ balls. ■

Proposed by Pierce Lai.

Countdown 9. Pierce is reading a light novel series. Suppose that, for this series, a chapter contains 20000 words, a book contains 2000 chapters, a volume contains 200 books, and the series contains 20 volumes. If the number of words in this series equals 4^a5^b for positive integers a, b, what is the value of $a+b$?

Solution. The number of words equals $2^4 10^{10}$, so the answer is $2+5+10 = \boxed{17}$. ■

Proposed by Pierce Lai.

Countdown 10. Suzie is buying buckets of chicken from Kungpao Furious Chicken. Suppose a small bucket contains 3 pieces of chicken, while a big bucket contains 7 pieces of chicken. If Suzie only buys these two bucket sizes, what is the least number of buckets she needs to buy to get exactly 100 pieces of chicken?

Solution. 100 is 2 mod 7, which can be obtained with 3 small buckets. Hence, the answer is 13 big buckets and 3 small buckets, for $\boxed{16}$ total buckets. ■

Proposed by Pierce Lai.

Countdown 11. Suppose $a \oplus b = \frac{ab}{a+b}$. What is the value of $((8 \oplus 6) \oplus 3) \oplus 2$?

Solution. Notice that $\frac{a+b}{ab} = \frac{1}{\frac{1}{a}+\frac{1}{b}}$. Hence, this equals $\frac{1}{\frac{1}{8}+\frac{1}{6}+\frac{1}{3}+\frac{1}{2}} = \frac{1}{\frac{1}{8}+1} = \boxed{\frac{8}{9}}$. ■

Proposed by Pierce Lai.

Countdown 12. An equilateral triangle and a regular hexagon have the same area. What is the ratio of the side length of the triangle to that of the hexagon? Express your answer in simplest radical form.

Solution. Notice that a hexagon can be divided into 6 equilateral triangles with the same side length. Hence, the answer is $\boxed{\sqrt{6}}$. ■

Proposed by Pierce Lai.

Countdown 13. A parallelogram has base length 5, short diagonal length 8, and long diagonal length 12. What is its area? Express your answer in simplest radical form.

Solution. Let d be the horizontal offset between the top and bottom sides of the parallelogram, and h the height of the parallelogram. We get that $(5-d)^2 + h^2 = 64$ and $(5+d)^2 + h^2 = 144$, or $20d = 80$ and thus $d = 4$. Therefore $h^2 = 63$ or $h = 3\sqrt{7}$, and so the area of the parallelogram is $\boxed{15\sqrt{7}}$. ■

Proposed by Pierce Lai.

Countdown 14. Suppose a cubic polynomial $P(x) = -2x^3 + ax^2 + bx + c$ has roots $-2, 3$, and -4. What is the value of $a + b + c$?

Solution. Notice that this implies that $P(x) = -2(x+2)(x-3)(x+4)$. Then, the sum of the coefficients is $-2 + a + b + c = P(1) = -2(1+2)(1-3)(1+4) = 60$, or $a + b + c = \boxed{62}$. ■

Proposed by Pierce Lai.

Countdown 15. There are 5 consecutive positive integers a, b, c, d, e, in that order, such that $a+b+c = d+e$. What is the smallest possible value of $c + d$?

Solution. We have $a + (a+1) + (a+2) = (a+3) + (a+4)$, so $3a + 3 = 2a + 7$, and thus $a = 4$. So $c = 6$, $d = 7$, and $c + d = \boxed{13}$. ■

Proposed by Swayam Gupta.

Countdown 16. How many four digit numbers (with no leading zero) satisfy the property that their digits are strictly increasing from left to right?

Solution. Notice that given a set of four distinct digits, there's only one way to order them in an increasing order. Hence, this is number of subsets of 4 digits from 1-9, so the answer is $\binom{9}{4} = \boxed{126}$. ■

Proposed by Pierce Lai.

Countdown 17. How many numbers between 1 and 1000 inclusive have digits which sum up to a number divisible by 5?

Solution. First, remove 1000 from the set and add 000 to the set. 1000 doesn't satisfy the condition but 000 does, so we'll need to subtract 1 at the end. In addition, write each number as three digits (e.g. $1 \to 001$) for easier counting. There are 100 choices for the first two digits, and 2 choices for whatever the third digit can be to satisfy divisibility by 5, so there are 200 possible numbers from 000 to 999. Then, we subtract 1 to get the answer $\boxed{199}$. ■

Proposed by Pierce Lai.

Countdown 18. Eda flips two fair coins, labeled Coin 1 and Coin 2. Given that at least one of the two coins landed heads, what is the probability that Coin 1 landed heads? Express your answer as a common fraction.

Solution. Without the information that at least one of the coins landed heads, there are four equally likely possibilities: HH, HT, TH, and TT. Of these, three have at least one heads: HH, HT, TH. Then, of these three possibilities, HH and HT landed heads. Hence, the answer is $\boxed{\frac{2}{3}}$. ■

Proposed by Pierce Lai.

Countdown 19. Suppose that the playlist of Pierce contains 1000 songs, which have average length 240 seconds each. However, when Pierce listens to the playlist of Pierce, Pierce does so at 4x speed. If Pierce listens to the playlist of Pierce twice, how many minutes does he spend listening?

Solution. Each song takes one minute on average at 4x speed, so twice the whole playlist is $\boxed{2000}$ minutes. ■

Proposed by Pierce Lai.

Countdown 20. Suppose $2x^2 + 4 = 14$. What is $2x^4 + 3x^2 + 1$?

Solution. We see that $x^2 = 5$, so $2x^4 + 3x^2 + 1 = 50 + 15 + 1 = \boxed{66}$. ■

Proposed by Pierce Lai.

Countdown 21. Find the positive integer n such that $1^2 + 4^2 + 6^2 + 4^2 + 1^2 + 2n^2 = 5!$.

Solution. We have $2n^2 + 70 = 120$, so $n = \boxed{5}$. ■

Proposed by Luke Robitaille.

Countdown 22. Suppose $a+b+c=0, 4a+2b+c=2$, and $9a+3b+c=0$. What is the value of $16a+4b+c$?

Solution. Notice that the coefficients of the given equations show up in the form $f(x)=ax^2+bx+c$. Then, we have $f(1)=0$, $f(2)=2$, and $f(3)=0$, so $f(x)=-2(x-1)(x-3)$. Therefore, the answer is $f(4)=-2(3)(1)=\boxed{-6}$. ■

Proposed by Pierce Lai.

Countdown 23. Lilith rolls a fair four-sided pyramid die (with faces labeled from 1-4) and a fair six-sided die (with faces labeled from 1-6), and multiplies the numbers on the bottom faces of the dice. What is the expected value of the number she gets? Express your answer as a common fraction.

Solution. The first number can be from 1 to 4, and the second number can be from 1 to 6. Hence, the answer is $\frac{1}{24}(1+2+3+4)(1+2+3+4+5+6)=\boxed{\frac{35}{4}}$. ■

Proposed by Pierce Lai.

Countdown 24. Borrow-mir and Steal-mir are playing rock-paper-scissors (RPS). In any given round of RPS, Borrow-mir has a $\frac{1}{3}$ chance of playing any one of the three moves, while Steal-mir has a 50% chance of playing scissors and an equal chance between the two other moves. What is the probability that Steal-mir wins a given round of RPS? Express your answer as a common fraction.

Solution. No matter what Steal-mir plays, Borrow-mir has a $\frac{1}{3}$ chance of playing the move that lets Steal-mir win, so the answer is $\boxed{\frac{1}{3}}$. ■

Proposed by Pierce Lai.

Countdown 25. Let r_1, r_2 be the solutions to the equation x^2+2x-2. What is the value of $r_1^2+r_2^2$?

Solution. By Vieta's, $r_1+r_2=-2$ and $r_1r_2=-2$. Hence, the answer is $(r_1+r_2)^2-2r_1r_2=\boxed{8}$. ■

Proposed by Pierce Lai.

Countdown 26. Lëa has built a snowman out of three perfect spheres of snow stacked on top of each other, all of different sizes, with the largest on bottom and the smallest on top. Suppose that the total height of the snowman is 72 inches, the radii of the snow spheres form an arithmetic sequence, and the total volume of the three spheres is 6912π cubic inches. What is the volume of the middle sphere?

Solution. Since the sphere radii form an arithmetic progression, the middle-widthed sphere must have diameter $72/3=24$ inches. Hence, the answer is $4/3*12^3\pi=\boxed{2304\pi}$ cubic inches. ■

Proposed by Pierce Lai.

Countdown 27. How many integers between 1001 and 2500 inclusive are divisible by 7 or 12 (or both)?

Solution. There are 215 numbers divisible by 7, 125 numbers divisible by 12, and 18 numbers divisible by 84, so the answer is $215+125-84=\boxed{255}$ ■

Proposed by Pierce Lai.

Countdown 28. Pierce is dunking quadruple-stuf Oreos in his favorite drink, soy sauce. Suppose he starts with one cup (240 milliliters) of 100% soy sauce, and every time he dunks an Oreo, a crumb of Oreo equal to half a milliliter is mixed into the drink. How many Oreos must he dunk so that the drink becomes a mixture of 40% Oreo and 60% soy sauce?

Solution. In total, there will be 240 milliliters of soy sauce and 160 milliliters of Oreo, so the answer is $\boxed{360}$. ∎

Proposed by Pierce Lai.

Countdown 29. What integer k satisfies the equation

$$4^{(4^4)} \times 8^{8\times 8} \times 16 = 2^k?$$

Solution. The $4^{(4^4)}$ term offers $2 * 4^4 = 512$ factors of 2, the 8^{8*8} term offers $3 * 64 = 192$ factors of 2, and the 16 term offers 4 factors of 2, so the answer is $512 + 192 + 4 = \boxed{708}$. ∎

Proposed by Pierce Lai.

Countdown 30. Thorin is throwing thoughts at a throne. Of these thrown thoughts, 40% are thorough while the others are not. 50% of the thrown thorough thoughts pass through the throne, though 30% of the thrown non-thorough thoughts do not pass through the throne. Of Thorin's thoughts that are thrown through the throne, what proportion are thorough thoughts? Express your answer as a common fraction.

Solution. $0.4 * 0.5 = 0.2$ of Thorin's thoughts are thorough and go through the throne, while $0.6 * 0.7 = 0.42$ of Thorin's thoughts are not thorough and go through the throne. Hence, the proportion is $0.2/(0.2+0.42) = \boxed{\frac{5}{31}}$. ∎

Proposed by Pierce Lai.

Countdown 31. Cola cans come in three sizes: 240 mL, 600 mL, and 1000 mL. What is the least number of cans of any size that in total sum up to exactly 2520 mL?

Solution. Notice that the 240 mL can is the only size with a nonzero tens digit. Hence, there must be at least 3 of those cans. Moreover, the one would need 5 240 cans to get a zero tens digit, but that would equal 1200 which can be obtained with fewer cans (2 * 600). Hence, there are exactly 3 240 mL cans, and the remaining amount is 1800 mL, which is obtained by 3 600 mL cans. Hence, the answer is $\boxed{6}$. ∎

Proposed by Pierce Lai.

Countdown 32. x, y, z are distinct positive integers such that $x+y+z = 13$, $x^2+y^2+z^2 = 65$, and $x^3+y^3+z^3 = 349$. What is the value of their product, xyz?

Solution. Notice that from the third equation, none of x, y, z are greater than 6, since $349 - 7^3 = 6$ and there aren't two distinct positive cubes that add up to 6. Moreover, one of them is 6 (since $3^3 + 4^3 + 5^3 = 216 < 349$) and one of them is 5 (since $3^3 + 4^3 + 6^3 = 307 < 349$), which means that the last one is 2. (We can verify that $2, 5, 6$ also satisfies the other two equations.) Hence, the answer is $\boxed{60}$. ∎

Proposed by Pierce Lai.

Countdown 33. &ndra rolls three dice. What is the probability that the product of their top faces is a prime? Express your answer as a common fraction.

Solution. The product is only a prime if two of the dice have ones on top and the last one is a prime. Having a 1 is $1/6$ probability and having a prime is $1/2$ probability, so the answer is $3(\frac{1}{6})^2(\frac{1}{2}) = \boxed{\frac{1}{24}}$. ■

Proposed by Pierce Lai.

Countdown 34. Turpen has a rectangular bar of chocolate. If the length of the bar of chocolate were increased by half its width and the width of bar were increased by half its length, its size would be increased by 150%. If the bar is longer than it is wide, what is the ratio of its width to its length? Express your answer as a common fraction.

Solution. Let the width be 1 and the length be r. (We can do this since scaling the bar up or down doesn't affect the problem's parameters.) Then, we get the equation $(r+1/2)(1+r/2) = 5r/2$. Multiplying out gives $2r^2+5r+2 = 10r$, or $2r^2-5r+2 = (2r-1)(r-2) = 0$. Since $r > 1$, $r = 2$, so the answer is $\frac{1}{r} = \boxed{\frac{1}{2}}$. ■

Proposed by Pierce Lai.

Countdown 35. Suppose $\sqrt{4-\sqrt{8-\sqrt{16-\sqrt{32-\sqrt{n}}}}} = \sqrt{2}$. What is the value of n?

Solution. We unwrap the equation to get $\sqrt{8-\sqrt{16-\sqrt{32-\sqrt{n}}}} = 2 \to \sqrt{16-\sqrt{32-\sqrt{n}}} = 4 \to \sqrt{32-\sqrt{n}} = 0 \to n = 32^2 = \boxed{1024}$. ■

Proposed by Pierce Lai.

Countdown 36. Dipper has a handful of tiny frogs, while Mabel has a handful of tiny toads. Suppose the average width of the tiny frogs Dipper is holding is 5mm and the average width of the tiny toads Mabel is holding is 10mm. In addition, suppose Dipper is holding 24 tiny frogs, and Mabel is holding 36 tiny toads. If Dipper and Mabel put together their tiny amphibians, what is the average width of all 60 tiny amphibians, in mm?

Solution. We have that the total width of the tiny frogs is $5*24 = 120$mm, and the total width of the tiny toads is $10*36 = 360$mm. Hence, the average width of all tiny amphibians is $(120+360)/60 = \boxed{8}$mm. ■

Proposed by Pierce Lai.

Countdown 37. Pierce is playing Boatknights. Suppose he begins playing Boatknights at 11am on Wednesday, and continuously plays until 3pm on Friday of the same week. How long does he play Boatknights, in minutes?

Solution. This is $4+2*24 = 52$ hours, or $\boxed{3120}$ minutes. ■

Proposed by Pierce Lai.
Remark: I need to sleep.

Countdown 38. Mai has a stack of n coins. First, she flips the top coin in her stack. If it is heads, she flips the next coin. If that is heads, she flips the coin after that, and so on, stopping either when sees a coin land tails for the first time, or when she has flipped all of the coins in her stack. What is the least possible value of n such that the expected number of heads is at least 0.99?

Solution. The expected number of heads is equal to $1 - \frac{1}{2^n}$, and the least n for which $\frac{1}{2^n} < 0.01$ is $\boxed{7}$. (As a side note, the expected number of heads and the expected number of tails are the same.) ■

Proposed by Pierce Lai.

Countdown 39. Nina and her dog Alexander are frolicking. Suppose that the number of times Nina has frolicked so far is twice the number of times Alexander would have frolicked if Alexander had frolicked four times less than currently, and that the number of times Alexander has frolicked is divisible by 5 plus the sum of the digits of the (base 10) number of times Nina has frolicked so far. (Furthermore, suppose that the two have frolicked a positive integer number of times.) What is the minimum number of times Nina has frolicked?

Solution. We see that, if N and A are the number of times Nina and Alexander have frolicked respectively, then $N = 2(A-4)$. Trying the first few smallest values for Alexander tells us that the least value which works is $A = 12$, giving $N = 16$, and 12 is divisible by $5+7 = 12$. Hence, the answer is $\boxed{16}$. ■

Proposed by Pierce Lai.

Countdown 40. Sdeu's Stupendous Stews sells tiny bowls of stew for 2 dollars, small bowls of stew for 3 dollars and large bowls of stew for 5 dollars. Finally, they also sell child-size bowls (so called as the bowl could fit an entire liquified child) of stew for 8 dollars. Suppose a group of three (distinguishable) people walk into the store, and each order a bowl of soup. If their total comes out to 13 dollars, how many different orders are possible?

Solution. A bit of counting tells us that the only possible orders are (2, 3, 8) and (3, 3, 5). Hence, the answer is $6+3 = \boxed{9}$. ■

Proposed by Pierce Lai.

Countdown 41. What is the smallest positive integer with exactly 18 positive divisors?

Solution. The sets of numbers which multiply to 18 are $(18), (9,2), (6,3), (3,3,2)$. Of these, $2^5 * 3^2 = 288$ and $2^2 3^2 5 = 180$ are smallest. Hence, the answer is $\boxed{180}$. ■

Proposed by Pierce Lai.

Countdown 42. Agony the ant starts at the origin of the coordinate plane, facing right. Then, he walks 10 units forward, turns clockwise 90 degrees, walks 10 units backwards, turns clockwise 180 degrees, walks forward 5 units, and then turns counterclockwise 270 degrees and walks 3 units forward. If he repeats this sequence of moves three more times, at what (x,y) coordinate will he end up?

Solution. Following the first set of moves, we see that he will end up at $(12, 15)$ after one repetition, facing right. Hence, after a total of four repetitions, he will end up at $\boxed{(48, 60)}$. ■

Proposed by Pierce Lai.

Countdown 43. Justin the First is selling apples. He has two types of apples, Red Not Very Delicious apples and Grainy Sherrywood apples, which he sells at fixed (positive) prices. Suppose his customer Justin the Second buys two Red Not Very Delicious apples and one Grainy Sherrywood apple, and his other customer Justin the Fifteenth buys one Red Not Very Delicious apple and three Grainy Sherrywood apples. If Justin the Fifteenth spends twice as much money as Justin the Second, what is the ratio of the price of a Red Not Very Delicious apple to the price of a Grainy Sherrywood apple? Express your answer as a common fraction.

Solution. Twice what Justin the Second bought is four Red apples and two Grainy apples, which should equal the price of one Red apple and three Grainy apples. Hence, the price of one Grainy apple equals the price of three Red apples, so the answer is $\boxed{1/3}$. ■

Proposed by Pierce Lai.

Countdown 44. Suppose x and y are positive integers. Given that $xy+x+y+1 = 54$ and $2xy+x+4y+2 = 110$, what is the value of xy?

Solution. The first equation tells us $(x+1)(y+1) = 54$ and the second gives $(x+2)(2y+1) = 110$. $2y+1$ is odd, and the odd factors of 110 are $1, 5, 11$, and 55. The first case is invalidated as y cannot be 0; the second gives $y = 2$ and $x = 20$, which fails the first equation; the third gives $y = 5$ and $x = 8$, which works; and the last gives $y = 27$ and $x = 0$, which fails as x must be positive. Hence, the answer is $5 * 8 = \boxed{40}$. ■

Proposed by Pierce Lai.

Countdown 45. Let f be a cubic polynomial such that $f(0) = 0$, $f(1) = 1$, $f(2) = 2$, and $f(3) = -3$. What is the value of $f(4)$?

Solution. From the first three conditions, we see that $f(x) - x$ must be a polynomial with roots $0, 1$, and 2, and thus $f(x) = x + cx(x-1)(x-2)$ for some constant c. $f(3) = -3$ tells us that $c = -1$, and therefore $f(4) = 4 - 4(3)(2) = \boxed{-20}$. ■

Proposed by Pierce Lai.

Countdown 46. If a, b are positive integers that satisfy $a^2b > 100$ and $ab^2 > 200$, what is the least possible value of a^3b^3?

Solution. Notice that a^3b^3 is a cube, with ab as the cube root. The smallest cube greater than 20000 is $28^3 = 21952$, but it doesn't work as $a = 4, b = 7$ gives $4 * 7^2 = 196 < 200$ for ab^2. 29 is prime so it doesn't work either. Next is $30^3 = 27000$, but it doesn't work since none of the pairs $(2,15), (3,10), (5,6)$ satisfy both inequalities. Therefore, the answer is $32^3 = \boxed{32768}$, with $a = 4, b = 8$. ■

Proposed by Pierce Lai.

Countdown 47. How many ways are there to permute the characters in "SORROW" such that no two identical characters are adjacent?

Solution. We use principle of inclusion-exclusion (PIE). There are two pairs of identical characters, O and R. There are $6!/2/2 = 180$ total permutations of "SORROW". For adjacent Os, we can pretend that the two Os are just one letter, and so there would be $5!/2 = 60$ permutations. Likewise for R. Then, for both adjacent, there are $4! = 24$ permutations. Hence, the answer is $180 - 60 - 60 + 24 = \boxed{84}$ ways. ■

Proposed by Pierce Lai.

Countdown 48. Suppose Joebob has 6 white sugar cubes, 8 gray sugar cubes, and 10 black sugar cubes, where cubes of the same color look exactly the same. He also has three cups of coffee in shades of red, blue, and green. How many ways can Joebob divide up all of the sugar cubes among his (distinguishable) cups?

Solution. The answer is just the product of three cases of "stars and bars". For example, consider the 6 sugar cubes. We imagine arranging the cubes in a line and adding two dividing bars, where sugar cubes before the first bar go to the cup with red coffee, sugar cubes between the bars go to the cup with green coffee, and sugar cubes after the second bar go to the cup with blue coffee. There are 8 objects, two of which are bars, so the number of ways to divide 6 indistinguishable cubes among three distinguishable cups is $\binom{8}{2} = 28$. Likewise, the gray cubes give $\binom{10}{2} = 45$ and the black ones $\binom{12}{2} = 66$, so the answer is $28 * 45 * 66 = \boxed{83160}$ ways. ■

Proposed by Pierce Lai.

Countdown 49. What is the 8th root of 78310985281, given that it is an integer?

Solution. First, since the root is an integer, we know it ends in $1,3,7$, or 9. In addition, $20^8 < 78310985281 < 30^8$, so it is either $21,23,27$, or 29. A brief check tells us that 78310985281 is not divisible by 3, so it is either 23 or 29. Finally, we note that 78310985281 is much closer to 20^8 than it is to 30^8, so we educatedly guess that the answer is $\boxed{23}$. ■

Proposed by Pierce Lai.

Countdown 50. Bob is frying some eggs. His skillet, however, has some issues. When he tries to fry an egg, the egg has a 40% chance of exploding and creating a universe, a 25% chance of imploding and destroying a universe, and a 35% chance of becoming a perfectly normal fried egg. Suppose that, when Bob has finished frying eggs, he ends up with 1001 perfectly normal fried eggs. What is the expected number of universes that were created, minus the expected number of universes that were destroyed?

Solution. We expect that Bob began with $1001/7 * 20 = 2860$ eggs, and so the difference is $2860 * (40\% - 25\%) = \boxed{429}$. ■

Proposed by Pierce Lai.

Countdown 51. Potato and Tomato are close friends. Suppose that one day Potato receives a bag of cream puffs, containing a positive integer number of cream puffs. Suppose that cream puffs are indivisible. Potato takes exactly one-fourth of the cream puffs, plus three cream puffs, from the bag. Tomato then takes exactly one-third of the cream puffs remaining, plus two cream puffs. Potato then takes exactly one-half of the cream puffs remaining, plus one cream puff. Tomato then takes the rest of the puffs, which is a prime number. What is the least number of puffs Potato could have taken?

Solution. The least prime is 2. Trying it as Tomato's final number of puffs gives the number before Potato's second turn as 6 cream puffs, the number before that 12 cream puffs, and the original number 20. Hence, the number Potato took was $8 + 4 = \boxed{12}$ puffs. ■

Proposed by Pierce Lai.

Countdown 52. Llanfairpwllgwyngyll is a town in Wales, whose train station is named Llanfairpwllgwyngyllgogerychwyrndrobwllllantysiliogogogoch. Suppose that, every day, the first train leaves this train station at 6am and that trains leave the station every x minutes afterward until the last train, where x is some positive integer. Suppose the last train to leave the station leaves at 10pm at night. What is the greatest positive integer n for which both n and $n+4$ would be valid values for x?

Solution. The period from 6am to 10pm spans 16 hours, or 960 minutes. The factors of 960 are $1,2,3,4,5,6,8,10,12,15,16,20,24,30,32$ and 960, so the answer is $\boxed{60}$. ■

Proposed by Pierce Lai.

Countdown 53. There are two threes in a sixteen-card deck. Josie draws three cards with replacement. What is the probability that at least two cards are threes? Express your answer as a common fraction.

Solution. There is an individual $\frac{1}{8}$ chance of getting a three. Then, split into cases. The probability that two cards are three is $\frac{1\cdot 1\cdot 8\cdot 3}{8^3}$ (remembering to multiply by three for various orderings), and the probability that all three cards are three is $\frac{1}{8^3}$. Hence, the answer is $\boxed{\dfrac{25}{512}}$. ■

Proposed by Josiah Kiok.

Countdown 54. Pierce has 75 bean buns. Suppose that he has m friends, such that if he tries to divide the 75 buns among his m friends equally, he is left with a prime number of buns over. If m is also prime, how many different values of m are there?

Solution. Clearly, m cannot be greater than 75, since otherwise we would have 75 buns left over, which is not prime. Moreover, m cannot be greater than 38 (except for 73), as otherwise each friend would only get one bun, and $75-m$ would be even and hence not prime. Checking the first few primes tells us that 7, 17, 29, 31, and 73 are the only ones which work, so the answer is $\boxed{5}$ values. ■

Proposed by Pierce Lai.

Countdown 55. Let T, A, I, L, O, and R be positive integers such that $T!+A!+I!+L!+O!+R! = 130$. What is the smallest possible value of $L+RATIO$?

Solution. The largest possible value of any of the letters is 5, since $6! = 720$ is too big. If none of the letters are equal to 5, then we consider how many are equal to 4. All six can't be equal to 4, or else their sum would be 144. If five are equal to 4, then the sixth factorial equals 10, which is impossible. And if at most four equal 4, then the largest possible value is $4!+4!+4!+4!+3!+3! = 108$ which is too small. So one of the letters (WLOG T) equals 5. Then $A!+I!+L!+O!+R! = 10$. Each of these letters is 1, 2, or 3. If one of them is 3, then the other four are 1. Otherwise, all five of them are 2. If all five are two, then $L+RATIO$ equals at least $5+2^5 = 37$. But if four are 1 and the fifth is 3, then $L+RATIO$ can be as small as $5+3=\boxed{8}$. ■

Proposed by Justin Time.

Countdown 56. Richard and Sally race across a 20 meter bridge while carrying heavy objects. Sally runs $\frac{3}{2}$ times as fast as Richard, though Sally only starts running once Richard has a 5 meter lead. When Sally catches up to Richard, Richard (in an effort to regain the lead) trips Sally (overkill issue), rendering her immobile. How many meters did Sally run?

Solution. In the time that Richard runs x meters, Sally would have run an additional $\frac{x}{2}$ meters (along with the x meters that Richard ran). Hence, Sally would close the distance between the two at half Richard's running rate.

Every time Richard runs 5 meters, the distance between them closes by $\frac{5}{2}$ meters. For the distance to close by 5 meters, Richard would need to run an additional 10 meters, so the answer is $\boxed{15}$. ■

Proposed by Josiah Kiok.

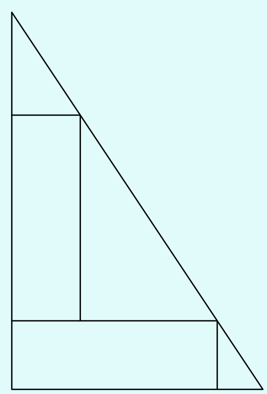
Countdown 57. Let $AUSTIN$ be a regular hexagon and let $MATH$ be a square. What is the square of the ratio of the area of $AUSTIN$ to the area of $MATH$? Express your answer as a common fraction.

Solution. WLOG let $AU = 1$. Then the area of the hexagon is $6 \cdot \frac{\sqrt{3}}{4} = \frac{3\sqrt{3}}{2}$ (by virtue of cutting the hexagon into six equilateral triangles). Noting that $AT = 2$ (again, by virtue of slicing the hexagon into triangles), the area of the square is 4.

The answer is then $\frac{\frac{27}{4}}{16} = \boxed{\frac{27}{64}}$. ■

Proposed by Josiah Kiok.

Countdown 58. Two 1×3 rectangles are placed inside a right triangle as shown below. What is the area of the right triangle? Express your answer as a common fraction.



Solution. By similarity, all the right triangles have side lengths in the ratio of 3 to 2. Thus the shorter side of the big triangle has length $3 + \frac{2}{3} = \frac{11}{3}$, and the longer side has length $\frac{11}{3} \cdot \frac{3}{2} = \frac{11}{2}$. So the answer is

$$\frac{1}{2} \cdot \frac{11}{3} \cdot \frac{11}{2} = \boxed{\frac{121}{12}}$$

■

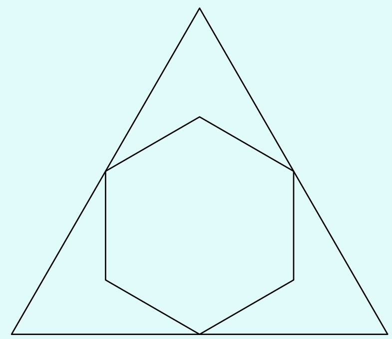
Proposed by Matthew Kroesche.

Countdown 59. Three circles of radius 3 are mutually externally tangent, and are all internally tangent to one large circle. What is the radius of the large circle? Express your answer in simplest radical form.

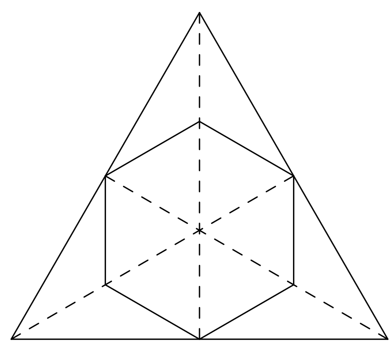
Solution. The centers of the three unit circles form an equilateral triangle with side length 6. Hence, the answer is $\boxed{3+2\sqrt{3}}$. ■

Proposed by Pierce Lai.

Countdown 60. A regular hexagon has three of its vertices at the midpoints of the three sides of an equilateral triangle, as shown in the figure. What is the ratio of the area of the hexagon to the area of the triangle? Express your answer as a common fraction.



Solution. Split the equilateral triangle up into six congruent right triangles. Each one has its hypotenuse bisected by the hexagon, so half its area is inside the hexagon and half is outside. Thus the answer is $\boxed{\frac{1}{2}}$.



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Proposed by Joshua Pate.

Challenge Problems

Challenge 1. Suppose I have a weighted coin, which has a p chance of flipping heads, where $p \leq 0.5$. Suppose that if I flip it twice, there is a 0.42 chance of getting one head and one tail. What is the probability of getting two heads from two flips? Express your answer as a decimal to the nearest hundredth.

Solution. We have that $2p(1-p) = 0.42$, or $p(1-p) = 0.21$, which by guess and check tells us that $p = 0.3$. Hence, the answer is $0.3^2 = \boxed{0.09}$. ■

Proposed by Pierce Lai.

Challenge 2. Chef Chris is making is a batch of his famed sodium-overdose Chrisserole casseroles. Suppose his batch contains 10 Chrisseroles and the number of cups of salt in each Chrisserole is an integer that ranges between 1 and 10 (inclusive). If Chris used a total of 59 cups of salt in the batch, what is the least possible value of the median of the number of cups of salt in each Chrisserole? Express your answer as a decimal to the nearest tenth.

Solution. Intuitively, if the median is m (and m is an integer), we get the maximum sum when the first 6 batches have m cups of salt each and the rest have 10 cups of salt each, which equals $6m + 40$ cups. Unfortunately, $59 > 6 * 3 + 40$, so the median is greater than 3. Thus, the answer is $\boxed{3.5}$, since one possible set with median 3.5 is $3, 3, 3, 3, 3, 4, 10, 10, 10, 10$. ■

Proposed by Pierce Lai.

Challenge 3. Max (short for Maxine) is playing a game with a pile of fair coins. Suppose she starts with 10 coins. In each round of the game, she flips each coin in her pile. If the coin lands heads, she removes the coin from the pile. If the coin lands tails, the coin duplicates, and she puts both coins back into the pile. Then, at the end of the round, Max adds a coin to her pile. At the end of the tenth round, what is the expected number of coins in her pile?

Solution. Notice that the flipping part of the game has net 0 effect on the expected value, since the coins are fair. Hence, the expected value is just the added coins, so the answer is $10 + 10 = \boxed{20}$. ■

Proposed by Pierce Lai.

Challenge 4. Walt Disney, a chemist, is cooking fried rice with his movie director friend, Walter White. Suppose the two each generate fried rice at a constant rate (don't ask how). If Walt takes 5 minutes to cook one plate's worth of fried rice and Walter takes 7 minutes to cook two plates' worth of fried rice, how long in minutes would it take for the two of them working together to cook three plates' worth of fried rice?

Solution. Walt cooks $\frac{1}{5}$ of a plate per minute, and Walter cooks $\frac{2}{7}$ of a plate per minute. Hence, the answer is $\frac{3}{\frac{1}{5}+\frac{2}{7}} = \frac{3}{\frac{17}{35}} = \boxed{\frac{105}{17}}$ minutes. ■

Proposed by Pierce Lai.

Challenge 5. Suppose that

$$\begin{array}{rcccc} & & A & B & S \\ \times & & & B & Y \\ \hline & B & A & B & Y \end{array}$$

where each letter represents a (not necessarily distinct) base ten digit. What is the greatest value of the four-digit number BAYS?

Solution. First, notice that A equals 1, since the thousands place of BABY is at least A times B (unless one of A or B is 0, but that would make BAYS smaller). Then, B also equals 1, since otherwise $AB0 * B0$ would be too big. Finally, we must have $(110+S)(10+Y) = 1110+Y$, or $10*S+110*Y+S*Y = 10$. Hence $Y = 0$ and $S = 1$, so the equation is simply $111*10 = 1110$. Thus, the answer is $\boxed{1101}$. ■

Proposed by Pierce Lai.

Challenge 6. Bowser is playing Celeste, a game in which he dies a lot. Each time he dies in the game, his fury grows. Suppose that his fury level starts at 0, and with each death his fury increases an amount proportional to the number of times he has died up to (and including) that death. (For example, his 10th death will increase his fury level by twice the amount of his 5th death.) Suppose that Bowser's 11th through 20th deaths (inclusive) increases his fury level by a total of 93. How much will his 21st through 40th deaths increase his fury level by?

Solution. Let X be the amount of fury from his first death. The amount of fury he has at his nth death is given by $pn(n+1)/2$ (the nth triangle number times some proportion constant). The fury obtained between his 11th through 20th deaths is thus the total fury at his 20th death minus the total fury at his 10th death, which is $p((21)(20)/2-(11)(10)/2) = 155p = 93$, so p must equal $3/5$. Hence, the fury obtained from his 21st through 40th deaths is given by $3/5*((40)(41)/2-(20)(21)/2) = \boxed{366}$. ■

Proposed by Pierce Lai.

Challenge 7. Matt and Jared are playing with a garbage bin. First, Matt plays for a random amount of time between 0 and 1 minutes, then Jared plays for a random amount of time between 0 and 1 minutes, and then Matt plays for a random amount of time between 0 and 1 minutes again. What is the probability that between the two of them, Matt gets more bin time? Express your answer as a common fraction.

Solution. This is geometric probability. Letting x, y, z represent the lengths of Matt's first time, Jared's time, and Matt's second time respectively, so that (x, y, z) is a uniformly random point in the unit cube. Then the probability that Matt gets more bin time is the region of the unit cube for which $x+z > y$. The region where $x+z \le y$ is a tetrahedron with vertices $(0,0,0), (0,1,0), (1,1,0), (0,1,1)$ which has area $\frac{1}{3}\cdot\frac{1}{2}\cdot 1 = \frac{1}{6}$, so the desired probability is $\boxed{\frac{5}{6}}$. ■

Proposed by Justin Xiao.

Challenge 8. Suppose m is a positive number which has 120 positive integer divisors. What is the greatest number of positive integer divisors m^3 can have?

Solution. Notice that we want the divisors to split up among as many primes as possible. (For example, suppose there were only 4 positive divisors. $2*3$ and 2^3 both have 4 divisors, but $2^3 3^3$ has 16 divisors while 2^9 only has 10.) Hence, since 120 equals $2^3 3*5$, the greatest possible number of divisors is $((3+1)^3(6+1)(12+1) = \boxed{5824}$. ■

Proposed by Pierce Lai.

Challenge 9. Motthew is watching the hit show "Breaking Brad's Bones". Suppose that the show has 8 episodes, and that the number of bones broken in each episode is a positive integer. Motthew notices that the number of bones broken in the first four episodes form an arithmetic sequence, the number of bones broken in the middle four episodes (episodes 3 to 6) form a geometric sequence, and the number of bones broken in the last four episodes form an arithmetic sequence. If the sum of the number of bones broken in episodes 3 and 6 equals 378, what is the maximum total number of bones broken across all 8 episodes?

Solution. Note that the constant sequence (189 bones for each episode) works, and gives a total of $189 * 8 = 1512$, but we can do a bit better. All other sequences are either strictly increasing or strictly decreasing. Since we can reverse any valid sequence to get another valid sequence, we can without loss of generality assume that otherwise the sequence is strictly increasing.

The 3rd and 6th terms (which add up to 378) are the first and last terms of the geometric series. The ratio of the geometric series must be a rational number $\frac{p}{q}$ in lowest terms. This means that the sum of the third term and the 6th term is divisible by $p^3 + q^3$ for some positive integers p, q.

Now, consider the possible ratios for p and q. The first four terms can be written as $a, a+r, a+2r, a+3r$, and the ratio must be $\frac{a+3r}{a+2r}$, for some positive integers a, r. Thus, the ratio must be strictly less than $3/2$ (but more than 1), which in particular means that $p > q \geq 3$.

Since $p^3 + q^3$ divides 378, this gives us only a few options for what p and q can be. The viable cubes are $27, 64, 125, 216$ and 343, and the sum of cubes must be a divisor of 378, or 378, 189, 126, 63, or 42. Checking all of these, we find that $189 = 64 + 125$ is the only viable solution. Thus, the only strictly increasing valid sequence is $64, 96, 128, 160, 200, 250, 300, 350$, whose sum is $189 * 6 + 350 + 64 = \boxed{1548}$. ■

Proposed by Pierce Lai.

Challenge 10. Math Minder has built 4 rovers that will explore the surface of Mars. Each rover weighs a positive integer number of pounds, and the sum of the reciprocals of the rovers' weights is exactly 1. Given that the largest of the rovers weighed n pounds, what is the largest possible value of n?

Solution. I (Matthew) will think about this some other time. By magic, the answer is $\boxed{42}$. ■

Proposed by Justin Xiao.

Challenge 11. A sphere is divided into pieces using five planes. What is the maximum possible number of pieces produced by the planes?

Solution. With the power of hamsters, the answer is $\boxed{26}$ pieces. ■

Proposed by Joshua Pate.

Challenge 12. Given a fixed 1:2:3 rectangular prism, in how many ways can three planes be placed so that each of the 12 midpoints of the sides of the prism is on at least one plane, and each plane contains at least three midpoints?

Solution. Apply an affine transformation so that the 1:2:3 prism just becomes a cube. First, we have to describe how many different types of planes there are that do this.

- First, there are the planes parallel to the faces of the cube. There are a total of **9** of these, six of which contain the six faces of the cube, and the remaining three of which lie midway between the pairs of opposite faces. Each of these planes contains four midpoints.

- Next, there are the planes parallel to one of the $1 \times \sqrt{2}$ rectangles formed by a pair of opposite edges of the cube. There are six such rectangles, and each one has two planes parallel to it – one on each side, for a total of **12**, each of which contains four midpoints. (Note that the plane containing the rectangle itself only contains two midpoints.)
- Next, there are the planes perpendicular to a space diagonal of the cube. There are four space diagonals, and three planes perpendicular to each one – two of which contain three midpoints each, and the third contains six (which are the vertices of a regular hexagon). This gives a total of **12** planes perpendicular to space diagonals.
- Next, there are the planes that contain the midpoints of two adjacent edges, and the midpoint of the edge connecting the two vertices other than the three formed by those two adjacent edges and the three directly above/below those two edges. This forms a triangle of side lengths $\frac{\sqrt{2}}{2}$, $\frac{\sqrt{6}}{2}$, and $\frac{\sqrt{6}}{2}$. There are **24** ways to choose such a triangle, and each gives rise to a plane that contains three midpoints.
- Finally, there are the planes that contain the midpoints of two opposite edges of the same face, and the midpoint of one edge of the opposite face that is not parallel to the first two edges. This forms a triangle of side lengths 1, $\frac{\sqrt{6}}{2}$, and $\frac{\sqrt{6}}{2}$. There are **24** ways to choose such a triangle, and each gives rise to a plane that contains three midpoints.

In total, there are $\binom{12}{3} = 220$ ways to choose three midpoints. The number of ways to choose three midpoints that lie in one of the planes we've described so far is

$$9\binom{4}{3} + 12\binom{4}{3} + 8\binom{3}{3} + 4\binom{6}{3} + 24\binom{3}{3} + 24\binom{3}{3} = 36 + 48 + 8 + 80 + 24 + 24 = 220$$

Thus, every plane we could possibly choose falls into one of these categories. We do casework based on how many points each plane contains.

- Suppose first that we choose three of the four six-midpoint planes. This covers all twelve midpoints, so this gives us **4** ways.
- Suppose that we only choose two of the four six-midpoint planes. There are six ways to make this choice. Regardless of which two we choose (even if they are consecutive versus opposite space diagonals) this covers exactly ten of the twelve midpoints, and the two that we miss are on opposite edges. Then for the third plane, we have to choose the one that passes through those two midpoints and is parallel to two faces of the cube. Choosing any other plane will cause it to contain six midpoints, which is a case we've already counted. This gives us **6** ways.
- Suppose that we choose exactly one of the four six-midpoint planes. Then the six midpoints that are excluded are the midpoints of the six edges connected to a pair of opposite vertices. We have to cover all six of these with just two planes. This means we have to cover three with one plane, and three with the other, since any plane that contains four of these six contains two other midpoints as well, and thus it's a case we've already counted. Then it just comes down to choosing which three we cover. It turns out that there are four ways to do this – one by putting the three edges neighboring each opposite vertex all together, and three by picking two of these edges and then one opposite edge to form a triangle with sides $\frac{\sqrt{2}}{2}$, $\frac{\sqrt{6}}{2}$, and $\frac{\sqrt{6}}{2}$. In total, this gives us **16** ways.
- Suppose we choose none of the four six-midpoint planes. Then we have to choose exactly three four-midpoint planes, and they cannot share any midpoints. This rules out all but the first two types in our list. (Note that in particular, none of the legal arrangements involve a $1 - \frac{\sqrt{6}}{2} - \frac{\sqrt{6}}{2}$ triangle.)
  - If one of our planes contains a face of the cube, then the other two planes have to be the two that are parallel to that one. Every other of the 18 planes we are allowed to choose from shares a midpoint with that one, except for four of the planes of the second type – but choosing one of these leaves four remaining vertices that are not coplanar. Thus there are **3** ways to do this – one for each pair of parallel faces of the cube.

- Otherwise, if one of our planes is midway between two faces of the cube (but the other two do *not* contain a face) then we have two ways to cover the remaining eight vertices. Either we could choose the other two planes that are midway between two faces (there is only **1** way to do this) or we could choose two of the four planes of the second type that are perpendicular to the one we have chosen and parallel to each other, for a total of $3 \times 2 =$ **6** ways.
- Finally, we cannot have all three of our planes be of the second type without having some two of them share a vertex. So this exhausts all the cases.

Thus the answer is $4+6+16+3+1+6 = \boxed{36}$ ways. ■

Proposed by Joshua Pate.

Tiebreaker Problems

> **Tiebreaker 1.** For what real number x does $\frac{\sqrt{x-3}+\sqrt{3}}{\sqrt{x+3}-\sqrt{3}}$ equal 3? Express your answer as a common fraction.

Solution. Multiplying out the denominator gives $\sqrt{x-3}+\sqrt{3} = 3\sqrt{x+3}-3\sqrt{3}$. Then, we add $3\sqrt{3}$ to both sides to get $\sqrt{x-3}+4\sqrt{3} = 3\sqrt{x+3}$. Squaring gets $x-3+8\sqrt{3x-9}+48 = 9x+27$, or $8\sqrt{3x-9} = 8x-18$, or $4\sqrt{3x-9} = 4x-9$. Squaring again gives $48x-144 = 16x^2-72x+81$, or $16x^2-120x+225 = 0$, or $(4x-15)^2 = 0$, so $x = \boxed{\dfrac{15}{4}}$. ■

Proposed by Pierce Lai.
Remark: Thanks to a friend for solving this.

> **Tiebreaker 2.** What is the largest integer k such that k^k divides 100 factorial?

Solution. Looking at the smallest primes, there are $50+25+12+6+3+1 = 97$ factors of 2, $33+11+3+1 = 48$ factors of 3, $20+4 = 24$ factors of 5, and 13 factors of 7. No larger prime works. We claim that 24 is the answer. First, note that 24 works, since $24^{24} = 2^{72}3^{24}$, which is less than the factors noted.

Then, we also need to show that it is the greatest. First, since the factors of every other prime in 100! is less than 25, k must be a product of 2s and 3s only. No value of k works for 3 3s (27^{27} has 81 factors of 3, which is too many). For 2 3s, the greatest value is $k = 18$; for 1 3, it's 24 (since 48 has too many 2s); and for 0 3s, it's 16. Hence, the answer is $k = \boxed{24}$. ■

Proposed by Pierce Lai.

> **Tiebreaker 3.** Nine students at a math competition order three pizzas, which cost \$16, \$16, and \$24 respectively. They order the three pizzas on three separate bills and tip \$2, \$4, and \$3. How much should each student pay, rounded to the nearest cent?

Solution. The total cost of all the pizzas is

$$16+16+24+2+4+3 = 65$$

so each student pays $\frac{65}{9} = 7.\overline{2}$ dollars. Thus we answer $\boxed{\$7.22}$. ■

Proposed by Josiah Kiok.